AAPT UNITED STATES PHYSICS TEAM
AIP 2020

2020 $F = ma$ Exam

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet. The only scratch paper you may use is scratch paper provided by the proctor. You may not use your own.
- This test contains 25 multiple choice questions. Select the answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily of the same level of difficulty.
- Correct answers will be awarded one point; incorrect answers or leaving an answer blank will be awarded zero points. There is no additional penalty for incorrect answers.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones cannot be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- The question booklet, the answer sheet, and all scratch paper will be collected at the end of this exam.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 1, 2020.**

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN!

We acknowledge the following people for their contributions to this year's exams (in alphabetical order):

Ariel Amir, JiaJia Dong, Mark Eichenlaub, Abijith Krishnan, Daniel Longenecker, Kye Shi, Brian Skinner, Paul Stanley, Mike Winer, and Kevin Zhou.

1. A ball is bouncing vertically between a floor and ceiling, which are both horizontal and separated by 4 m. All collisions are perfectly elastic, and when the ball hits the floor, it has a speed of 12 m/s. How long does a complete up-down cycle take?

   (A) 0.3 s
   (B) 0.4 s
   (C) 0.6 s
   (D) 0.8 s ← **CORRECT**
   (E) 2.4 s

Solution

If the ball begins at the floor and travels towards, the ceiling, its height h above the floor at a time t is given by

$$h = (12\text{ m/s})t - \frac{1}{2}gt^2.$$

To find the time for the ball to reach the ceiling, we substitute in $h = 4$ m and $g = 10\text{ m/s}^2$. Solving the quadratic equation for t, we obtain

$$t_{\text{top}} = 0.4\,\text{s}.$$

By symmetry, the time for the ball to go down from the ceiling to the floor is the same as the time for the ball to go from the floor to the ceiling. This means the answer is 2×0.4 s $= 0.8$ s.

2. A uniform rod of mass m and length ℓ has moment of inertia $m\ell^2/12$ about an axis that is perpendicular to the rod and passes through its center. What is the moment of inertia of a uniform square plate with mass M and side length L about the axis along its diagonal?

   (A) $ML^2/12$ ← **CORRECT**
   (B) $\sqrt{2}\,ML^2/12$
   (C) $ML^2/6$
   (D) $ML^2/4$
   (E) $ML^2/3$

Solution

Place the origin at the center of mass of the square. The perpendicular axis theorem states that for a two-dimensional mass distribution in the x-y plane,

$$I_z = I_x + I_y.$$

If the x and y axes are oriented parallel to the square's sides, then the moments of inertia I_x and I_y are both those of a uniform rod of mass M and length L,

$$I_x = I_y = \frac{ML^2}{12}.$$

This implies

$$I_z = \frac{ML^2}{6}.$$

Now consider rotating these axes by 45° about the z axis, giving axes x' and y' along the diagonals of the square. The perpendicular axis theorem now gives

$$I_z = I_x' + I_y'$$

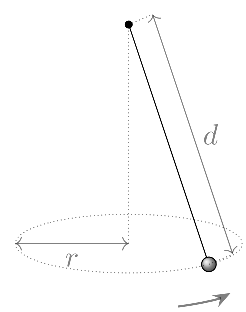
where I_x' is the desired answer. By symmetry, $I_x' = I_y'$, so

$$\frac{ML^2}{6} = 2I_x'$$

and

$$I_x' = \frac{ML^2}{12}.$$

3. A conical pendulum of length d swings in a horizontal circle of radius r, as shown. If ω is the angular frequency of this motion, what is ω^2?



(A) g/d
(B) g/r
(C) $g/\sqrt{d^2 + r^2}$
(D) $g/\sqrt{d^2 - r^2}$ ← **CORRECT**
(E) $g/\sqrt{dr}$

Solution

The vertical component of tension must cancel the force of gravity. The horizontal component of tension must provide the centripetal force.

Let θ be the angle of the pendulum from vertical, satisfying

$$\tan\theta = \frac{r}{\sqrt{r^2 - d^2}}.$$

For the vertical component of tension to cancel gravity, we must have

$$mg = T\cos\theta.$$

For the horizontal component of tension to equal the centripetal force, we have

$$m\omega^2 r = T\sin\theta.$$

Dividing these equations gives

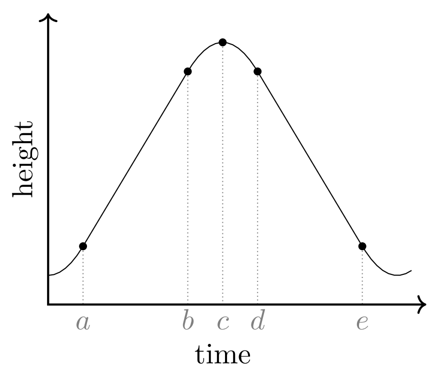
$$\frac{\omega^2 r}{g} = \tan\theta.$$

Substituting in our result for $\tan\theta$ and solving for ω^2, we obtain

$$\omega^2 = \frac{g\tan\theta}{r} = \frac{g}{\sqrt{d^2 - r^2}}.$$

Notice that the in the limit $r \to d$ the pendulum is horizontal, which requires and infinite angular frequency. In the limit $r \to 0$ we obtain the ordinary formula for the angular frequency of a pendulum in a plane. Thus, the problem could also have been solved using just these limiting cases.

4. In "zero-g" airplane rides, passengers can float around the cabin, as if they were weightless. A flight trajectory for such a ride is shown below, with a few points during the journey labelled. Which of the following statements is correct? Take $\mathbf{a}$ to be the acceleration of the plane during the "zero-g" part of the flight.



(A) The "zero-g" flight begins at a and ends at c, during which $|\mathbf{a}| = g$ and $\mathbf{a}$ points up.

(B) The "zero-g" flight begins at a and ends at e, during which $|\mathbf{a}| = 0$.

(C) The "zero-g" flight begins at b and ends at d, during which $|\mathbf{a}| = g$ and $\mathbf{a}$ points down. ← **CORRECT**

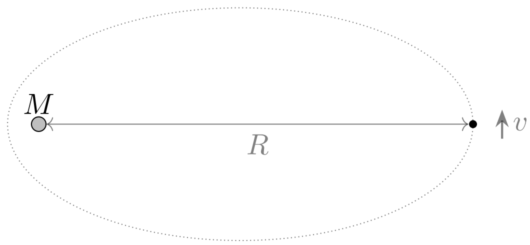
(D) The "zero-g" flight begins at c and ends at e, during which $|\mathbf{a}| = g$ and $\mathbf{a}$ points down.

(E) The "zero-g" flight begins at d and ends at e, during which $|\mathbf{a}| = 0$.

Solution

The zero-g segment corresponds to the part where the airplane falls freely with the passengers inside. That is, during the zero-g segment the acceleration is g downward, and hence a concave down parabola, corresponding to the segment from b to d.

5. A mote of dust is initially located at distance R from the sun, which has mass M. At this point, the mote has a small tangential velocity v. Which of the following is a good approximation for the distance of closest subsequent approach between the mote and the sun?



(A) $\dfrac{R^3v^4}{G^2M^2}$

(B) $\dfrac{R^3v^4}{2G^2M^2}$

(C) $\dfrac{R^2v^2}{2GM}$ ← **CORRECT**

(D) $\dfrac{R^2v^2}{GM}$

(E) $\dfrac{2R^2v^2}{GM}$

Solution

If the particle's closest approach to the sun is R' and its speed there is v', then angular momentum conservation tells us

$$mvR = mv'R'.$$

The particle's initial kinetic energy is negligible because v is small, and its initial potential energy is negligible because it is very far from the sun compared to its closest approach. So we can set the particle's initial energy to zero, and by energy conservation,

$$0 = -\frac{GMm}{R'} + \frac{1}{2}mv'^2.$$

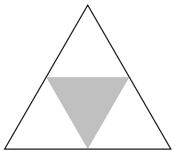
Plugging in our expression for v' from angular momentum conservation gives

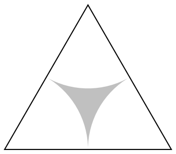
$$\frac{GMm}{R'} = \frac{1}{2}\frac{mv^2R^2}{R'^2}.$$

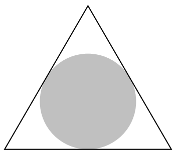
Solving for R' gives the answer,

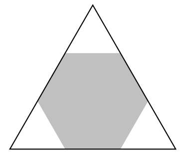
$$R' = \frac{R^2v^2}{2GM}.$$

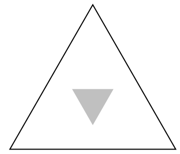
6. A three-legged table is shaped like a uniform equilateral triangle, and has identical legs at each corner. When the mass of an object placed on the center of the table exceeds m_{max}, the table's legs will all simultaneously break. Which of the following shaded regions shows the area within which an object of $2m_{\text{max}}/3$ can be placed without breaking any legs of the table?

(A)  ← **CORRECT**

(B) 

(C) 

(D) 

(E) 

Solution

A leg will break if there is a force of at least $m_{\text{max}}g/3$ on it, i.e. if it bears more the half of the weight of the mass $2m_{\text{max}}/3$.

Label the vertices of the triangle A, B, and C, and consider torque balance about an axis parallel to BC and passing through the mass.

The torque from legs B and C is $y(F_B + F_c)$, with y the distance of the mass from the segment BC. The torque from leg A is $(h - y)F_A$, with h the height of the triangle. Balancing these torques,

$$y(F_B + F_C) = (h - y)F_A.$$

Additionally,

$$F_B + F_C = \frac{2}{3}m_{\text{max}}g - F_A.$$

Combining the previous two equations and solving for F_A,

$$F_A = \frac{2}{3}mg\frac{y}{h}.$$

This shows that the leg at A supports more than half the weight if the distance from the mass to BC is more than half the height of the triangle. The same logic holds for the other two sides. Combining these constraints gives choice (A).

7. Two satellites are initially in identical circular orbits around the Sun, with orbital speed $v = 1 \times 10^4\,\text{m/s}$. The first satellite fires its thrusters toward the Sun, and quickly obtains a radial velocity of $\Delta v_r = 1\,\text{m/s}$. The second satellite instead fires its thrusters behind it, and quickly increases its tangential velocity by Δv_t. If the two satellites subsequently perform orbits with the same period, approximately what was Δv_t?

   (A) $0.000\,05\,\text{m/s}$ ← **CORRECT**

   (B) $0.005\,\text{m/s}$

   (C) $0.5\,\text{m/s}$

   (D) $1\,\text{m/s}$

(E) 50 m/s

Solution

By Kepler's third law, the period only depends on the semimajor axis a. However, the total energy of an elliptic orbit is determined by the semimajor axis by $E = -GMm/2a$. Hence the two satellites must have the same energy, and hence the same final speed, so

$$(v + \Delta v_t)^2 = v^2 + (\Delta v_r)^2$$

which gives

$$\Delta v_t \approx \frac{(\Delta v_r)^2}{2v} = 5 \times 10^{-5}\,\mathrm{m/s}.$$

8. A conveyor belt is moving with velocity v to the east. A block with velocity v to the south slides from the ground onto the conveyor belt. The coefficient of friction between the block and the belt is μ. The block stops slipping after a time

(A) $\dfrac{v}{\sqrt{2}\,\mu g}$

(B) $\dfrac{v}{\mu g}$

(C) $\dfrac{\sqrt{2}\,v}{\mu g}$ ← **CORRECT**

(D) $\dfrac{2v}{\mu g}$

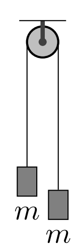
(E) The block never stops slipping.

Solution

Work in the reference frame of the conveyor belt. In this frame, the block's initial velocity perpendicular to the edge of the belt is v and its initial velocity along the belt is also v. Because these are at right angles, the initial speed of the block is $\sqrt{v^2 + v^2} = \sqrt{2}v$. The frictional acceleration is μg, so the time to come to a stop is $\sqrt{2}v/\mu g$.

The following information is relevant to problems 9 and 10.

9. Two equal masses m are connected by an elastic string that acts like an ideal spring with spring constant k and unstretched length l. The two masses are hung over a frictionless pulley. What is the total length of the string at equilibrium? (Diagram not necessarily to scale.)



(A) $2mg/k$
(B) $l + mg/k$ ← **CORRECT**
(C) $l + mg/2k$
(D) $l + 2mg/k$
(E) There is not enough information to decide.

Solution

Balancing the string tension with gravitational force on either of the masses gives

$$kx = mg,$$

so the total length is

$$l + x = l + \frac{mg}{k}.$$

10. The two masses are both displaced downward by a small vertical distance x and simultaneously released from rest. What is the period of oscillation?

(A) $2\pi\sqrt{l/g}$
(B) $\pi\sqrt{2m/k}$ ← **CORRECT**
(C) $2\pi\sqrt{m/k}$
(D) $2\pi\sqrt{m/k + l/g}$
(E) There is not enough information to decide.

Solution

Gravity does not affect the oscillation frequency, so we may treat the two-body spring-mass system as though it is isolated (we can also write down equations of motion to confirm this assumption by considering displacement from the equilibrium positions).

The reduced mass is $\mu = \frac{m}{2}$, so the oscillation frequency is

$$2\pi\sqrt{\frac{\mu}{k}} = \pi\sqrt{\frac{2m}{k}}.$$

Equivalently, one can notice that the length of the string is changed by $2x$ when each mass is displaced downward by x, so that each mass experiences a restoring force

$$F = -2kx.$$

In other words, each mass experiences an effective spring constant $k_{\text{eff}} = 2k$. The oscillation period

$$T = 2\pi\sqrt{\frac{m}{k_{\text{eff}}}} = \pi\sqrt{\frac{2m}{k}}.$$

11. An escalator can carry passengers up a vertical distance of 10 m in 30 s. A mischievous person of mass 50 kg walks down the up-escalator so that they stay in place with respect to the building. If the child does this for 30 s, the total work the child performs on the escalator, in the frame of the building, is

    (A) -10^4 J
    (B) -5×10^3 J $\leftarrow$ **CORRECT**
    (C) 0 J
    (D) 5×10^3 J
    (E) 10^4 J

Solution

The average power is $\bar{F}v_y$ where $\bar{F} = -mg$ is the average force exerted by the child on the escalator, and v_y is the vertical velocity of the point of contact, i.e. one or both of the child's shoes, when they are touching the escalator. This velocity is precisely the speed of the escalator, $v_y = 10/30$ m/s. The result is hence $(-500)(1/3)(30)$ J $= -5 \times 10^3$ J. Notice that the work done by the child on the escalator is the negative of the work done by the escalator on the child, which has the same value $W = mgh$ as it would have if the child had remained stationary. (Here h is the height of the escalator.)

One might think that the answer is 0 J because the center of mass of the child isn't moving. This is incorrect because work depends on the motion of the part of a system the force is applied on, not the motion of the system's center of mass. Another incorrect way to get at an answer of 0 J is to work in the reference frame moving upward with the escalator. In this frame the escalator behaves like a static set of stairs, so it can do no work on the child. That is correct, but the amount of work done depends on the reference frame, and the problem asked for the work done in the frame of the building.

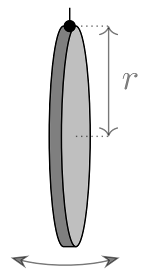
12. A platform juts out horizontally from the edge of a building. If the platform is modeled as a uniform metal rod, which one of the following statements about the tensile and compressive stress is correct?

    (A) There is a horizontal compression throughout the rod.
    (B) There is a horizontal tension throughout the rod.
    (C) There is horizontal tension in the top of the rod and a compression in the bottom. $\leftarrow$ **CORRECT**
    (D) There is a horizontal tension near the middle of the rod and a compression near the end.
    (E) There is a horizontal compression near the middle of the rod and a tension near the end.

Solution

There must be tensive and compressive forces at the top and bottom of the rod, respectively, to balance the torque on the rod from gravity; ultimately these forces come from the wall the rod is attached to.

13. A uniform disk of mass m and radius r is attached at its edge to a flexible pivot on the ceiling. It is given a small displacement *perpendicular* to the plane of the disk, so that it begins to oscillate perpendicular to the plane of the disk. What is the period of oscillation? The moment of inertia of a disk about the axis going through its center and perpendicular to the plane it's in is $I_{\text{disk}} = \frac{1}{2}mr^2$



(A) $\pi\sqrt{2r/5g}$

(B) $\pi\sqrt{5r/g}$ ← **CORRECT**

(C) $\pi\sqrt{6r/g}$

(D) $2\pi\sqrt{r/g}$

(E) $2\pi\sqrt{2r/g}$

Solution

Let $I_\perp = \frac{1}{2}mr^2$ be the moment of inertia about the central axis of the disk, let I_{cm} be the moment of inertia of the disk about an axis through the center of the disk *parallel* to the plane of the disk, and let I denote the moment of inertia about the axis of oscillation. By the Perpendicular Axis theorem,

$$I_\perp = I_{\text{cm}} + I_{\text{cm}},$$

so

$$I_{\text{cm}} = \frac{1}{2}I_\perp = \frac{1}{4}mr^2.$$

By the Parallel Axis theorem,

$$I = I_{\text{cm}} + mr^2 = \frac{5}{4}mr^2..$$

For some small oscillation displacement by an angle θ, the torque about the pivot is

$$\tau = -mgr\sin\theta \approx -mgr\theta,$$

so applying $\tau = I\ddot{\theta}$ gives

$$\ddot{\theta} = \frac{\tau}{I} = -\frac{4}{5}\frac{g}{r}\cdot\theta = -\omega^2\theta.$$

Thus the period of oscillation is

$$T = \frac{2\pi}{\omega} = \pi\sqrt{5\cdot\frac{r}{g}}.$$

14. A large block of mass 5 kg is moving to the right at a velocity of 10 m/s. Spaced out every meter are smaller, initially stationary blocks of mass 1 kg. All collisions are perfectly inelastic. Neglecting friction, how far will the large block travel before its velocity has decreased to below 3 m/s?

(A) 5 m

(B) 8 m

(C) 12 m ← **CORRECT**

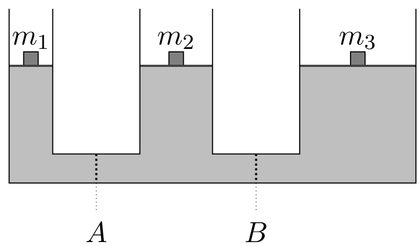
(D) 17 m

(E) 50 m

Solution

Momentum is conserved, and the block starts out with 50 kg m/s of momentum. For its velocity to go below 3 m/s, its mass must exceed 16.7 kg. The first time this happens is after it absorbs 12 blocks and has mass 5 kg + 12 kg = 17 kg

15. In the device shown, blocks of various masses are placed on pistons so that the device is in equilibrium. (The fluid in the drawing is to scale.) There are valves that are both initially open at locations A and B. One of the valves is closed, and the system is allowed to come to equilibrium. How will this affect the height of the mass m_2?



(A) Closing either valve will have no effect on the height of m_2. ← **CORRECT**

(B) Closing either valve causes m_2 to rise.

(C) Closing either valve causes m_2 to fall.

(D) Closing valve A causes m_2 to rise, and closing valve B causes m_2 to fall.

(E) Closing valve A causes m_2 to fall, and closing valve B causes m_2 to rise.

Solution

The gauge pressure of the water directly under block m_1 must be $\frac{m_1 g}{A_1}$, where A_1 is the cross-sectional area of the tube containing block m_1. The pressure in the horizontal section of the tube is therefore $\rho g h + \frac{m_1 g}{A_1}$. But by the same reasoning, the pressure is also $\rho g h + \frac{m_2 g}{A_2}$ and $\rho g h + \frac{m_3 g}{A_3}$. We conclude $\frac{m_1}{A_1} = \frac{m_2}{A_2} = \frac{m_3}{A_3}$. This is the condition for equilibrium in the device. It we close valve A, the condition for equilibrium in the right two tubes is $\frac{m_2}{A_2} = \frac{m_3}{A_3}$, which is already met, so these tubes remain in equilibrium. The same applies to valve B. In other words, the pressure is the same to the left and right of a valve, so closing the valve doesn't affect the equilibrium.

The following information is relevant to problems 16 and 17.

16. A mass M sits on top of a vertical spring of spring constant k, in equilibrium. A mass m is held a height h above it. The mass M is then pushed downward by a distance Δx, and both masses are released from rest simultaneously. For what value of h will the two masses first collide when M first returns to its equilibrium position?

(A) $\dfrac{\pi M^2 g^2}{4k^2 \Delta x}$

(B) $\dfrac{8Mg}{\pi^2 k}$

(C) $\dfrac{Mg}{k}$

(D) $\dfrac{\pi^2 Mg}{8k}$ ← **CORRECT**

(E) $\dfrac{\pi k \Delta x^2}{4Mg}$

Solution

For the masses to collide at the equilibrium position of M, we need for the time during which m is falling to be equal to $\frac{\pi}{2}\sqrt{\frac{M}{k}}$, which is one quarter of the oscillation period of the spring. The time to free fall a distance h, starting from rest, is given by $\sqrt{\frac{2h}{g}}$, so we have

$$h = \frac{\pi^2 Mg}{8k}.$$

17. Assume that h takes the value found in the previous question, and that the collision between the two masses is perfectly elastic. For what value of Δx will m rebound to a maximum height that is exactly equal to its original height?

(A) $\dfrac{\pi g}{2k}\sqrt{\dfrac{m^3}{M}}$

(B) $\dfrac{\pi g}{k}\sqrt{\dfrac{m^3}{2M}}$

(C) $\dfrac{2g}{\pi k}\sqrt{\dfrac{2m^3}{M}}$

(D) $\dfrac{4mg}{\pi k}$

(E) $\dfrac{\pi mg}{2k}$ ← **CORRECT**

Solution

Since the potential energy of the compressed spring is turned into kinetic energy for mass M, such that $\frac{1}{2}Mv_M^2 = \frac{1}{2}k(\Delta x)^2$, we have

$$v_M = \sqrt{\frac{k}{M}}\Delta x,$$

where v_M is the speed of mass M just before colliding. The speed of mass m just before colliding is

$$v_m = \sqrt{2gh} = \frac{\pi g}{2}\sqrt{\frac{M}{k}}.$$

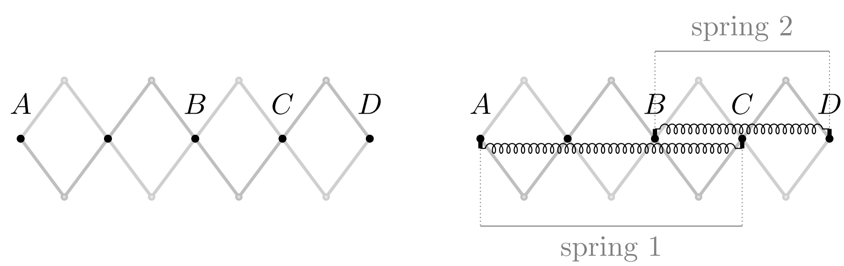
We need $mv_m = Mv_M$ for an elastic collision in which the masses just turn around at the same speed with which they came in. This equation gives

$$\frac{\pi mg}{2}\sqrt{\frac{M}{k}} = M\sqrt{\frac{k}{M}}\Delta x,$$

so that

$$\Delta x = \frac{\pi mg}{2k}.$$

18. An extendable arm is made from rigid beams free to pivot around the dots shown. Spring 1, with equilibrium length L_1, is attached between points A and C while spring 2, with equilibrium length L_2 is attached between B and D. The system is allowed to come to equilibrium. In equilibrium, what is the ratio of the tension in spring 1 to the tension in spring 2?



(A) $-2/3$ ← **CORRECT**
(B) $-3/4$
(C) -1
(D) $-3/2$
(E) There is not enough information to determine the ratio.

Solution

Imagine extending the entire arm by an amount $\mathrm{d}x$. Then spring 1 gets longer by $\frac{3}{4}\mathrm{d}x$ and spring 2 by $\frac{1}{2}\mathrm{d}x$. The change in potential energy of the arm is $T_1\frac{3}{4}\mathrm{d}x+T_2\frac{1}{2}\mathrm{d}x$. Because the arm is in equilibrium, small extensions do not change the potential energy for very small $\mathrm{d}x$, so $T_1\frac{3}{4}\mathrm{d}x+T_2\frac{1}{2}\mathrm{d}x = 0$. Solving for $\frac{T_1}{T_2}$, we get $\frac{T_1}{T_2} = -\frac{2}{3}$.

19. Consider an axially symmetric object that experiences no external forces and initially rotates about its symmetry axis. The object then changes its shape while remaining axially symmetric. Afterward, it is found that its moment of inertia about the symmetry axis has increased. How have its kinetic energy T and angular velocity ω changed?

(A) T has decreased, and ω has decreased. ← **CORRECT**

(B) T has decreased, and ω stays the same.

(C) T stays the same, and ω has decreased.

(D) T has increased, and ω has increased.

(E) T has increased, and ω stays the same.

Solution

At every moment we have $L = I\omega$, and the angular momentum L is conserved. The kinetic energy can be written in terms of L and I,

$$T = \frac{I\omega^2}{2} = \frac{L^2}{2I}.$$

Since I increases, T decreases. Since $L = I\omega$ is conserved, since I increases, ω decreases.

20. A well-calibrated scale reads zero when nothing is placed on it. When a deflated balloon is placed on the scale, it reads mg. When a non-airtight box is placed on the scale, it reads Mg, where $M > m$. The balloon is inflated with helium, so that it floats upward in air. The balloon is placed in the box, and the box is placed on the scale. If the scale reads W, which of the following is true?

(A) $W \leq (M - m)g$

(B) $(M - m)g < W < (M + m)g$

(C) $W = (M + m)g$

(D) $W > (M + m)g$

(E) None of the above are necessarily true. $\leftarrow$ **CORRECT**

Solution

Consider the forces on the box. Before the balloon is in the box, the weight of the box, the buoyant force on the box due to the atmosphere, and the normal force from the scale all balance, and the normal force is Mg. When the balloon is placed in, the only change in the forces on the box is an upward force, as the balloon pushes up on the top of the box. Thus the answer must be less than Mg. However, we don't know whether it is larger or smaller than $(M - m)g$ because this depends on how buoyant the balloon is, so there is not enough information to decide.

21. A person stands on the seat of a swing and squats down, so that the distance between their center of mass (CM) and the swing's pivot is ℓ. As the swing gets to the lowest point, the speed of their CM is v. At this moment, they quickly stand up, and thus decrease the distance from their CM to the swing's pivot to ℓ'. Immediately after they finish standing up, their CM speed is v'. Which of the following statements is correct? You may neglect friction, the change in moment of inertia of the person about their CM, and the time taken to stand up.

(A) $v/\ell = v'/\ell'$

(B) $v = v'$

(C) $v\ell = v'\ell'$ $\leftarrow$ **CORRECT**

(D) $\frac{1}{2}v^2 = \frac{1}{2}v'^2 + g(\ell - \ell')$

(E) Multiple statements are correct.

Solution

Because the person is doing work to stand up, the energy is *not* conserved in this problem. Instead, the angular momentum about the swing pivot is conserved since there is no net torque. The angular momentum about the person's CM is $mv\ell$ while squatting, and $mv'\ell'$ after they stand up. (The linear momentum is not conserved, because internal forces in the swing give the person an impulse as they stand up.)

22. A point mass m sits on a long block, also of mass m, which rests on the floor. The coefficient of static and kinetic friction between the mass and the block is μ, and the coefficient of static and kinetic friction between the block and the floor is $\mu/3$. An impulse gives a horizontal momentum p to the point mass. After a long time, how far has the point mass moved relative to the block? Assume the mass does not fall off the block.

(A) $\dfrac{3p^2}{8m^2\mu g}$ ← **CORRECT**

(B) $\dfrac{15p^2}{32m^2\mu g}$

(C) $\dfrac{9p^2}{16m^2\mu g}$

(D) $\dfrac{3p^2}{10m^2\mu g}$

(E) $\dfrac{3p^2}{4m^2\mu g}$

Solution

Initially, the mass will begin to slip on the block, which in turn will begin to slip on the floor. The frictional force between the point mass and the block is $-\mu mg$, and the frictional force between the block and the floor is $(\mu/3)(2m)g$. Because $\mu > 2\mu/3$, the block will always be moving forward during the point mass's motion. Applying $f = \mu N$, the mass has a backward acceleration of μg, while the block has a forward acceleration of $(\mu - 2\mu/3)g = \mu g/3$. Therefore, the relative acceleration is $a_{\text{rel}} = (4/3)\mu g$.

The initial relative velocity of the blocks is $v_i = p/m$, so slipping between the point mass and block stops after a time $t = v_i/a_{\text{rel}}$. The average relative velocity during this time is $v_i/2$. Therefore, the total relative motion is $v_i^2/2a_{\text{rel}}$, giving an answer of $(3/8)p^2/(m^2\mu g)$.

23. Assume that the drag force for a fish in water depends only on the typical length scale of the fish R, its velocity v, and the density of water ρ. A pufferfish is about 10 cm in length and swims at about 5 m/s. How fast does a clown fish, about 1 cm in length, need to swim such that it experiences the same drag force as the pufferfish?

(A) 5 m/s

(B) 16 m/s

(C) 50 m/s ← **CORRECT**

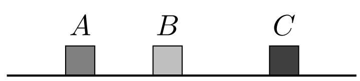
(D) 500 m/s

(E) 2500 m/s

Solution

Dimensional analysis gives $F \propto R^2 v^2 \rho$. For the two fish to have the same drag, $R_{\text{puff}}^2 v_{\text{puff}}^2 = R_{\text{clown}}^2 v_{\text{clown}}^2$, giving the correct answer 50 m/s.

24. Three boxes A, B and C lie along a straight line on a horizontal frictionless surface, as shown. Box A is initially moving to the right with speed v while the other two boxes are initially at rest. If all collisions are elastic, and the masses of the boxes can be chosen freely, which of the following is closest to the maximum possible final speed of box C?



(A) v

(B) $2v$

(C) $3v$

(D) $4v$ ← **CORRECT**

(E) $5v$

Solution

Consider the elastic collision between A and B. After the collision, box B's speed is

$$v_B = \frac{m_A - m_B}{m_A + m_B} v + v \leq 2v.$$

So if $m_C \ll m_B \ll m_A$, repeating this reasoning shows that the final speed for box C will be $4v$. This is the maximum possible speed.

This is the intended physical intuition to answer the question, but showing it more rigorously is subtle. It's clearly optimal to have m_A as large as possible. However, box C can be hit multiple times if the ratio m_C/m_B is made higher, with each impact transferring more momentum. Thus, one might suspect that having a larger value of m_C/m_B could yield a higher final speed.

This is in fact not the case, because when m_C/m_B is larger enough to have multiple collisions, the effect of each collision becomes much smaller. For example, consider the case $m_C/m_B = 1$. In this case three collisions happen in total, and boxes B and C end up with the same final speed $2v$. Therefore, if we make m_C/m_B slightly higher than 1, box B will be able to catch up with box C, hitting it a second time. However, this gives a final speed of just over $2v$, which is much worse than $4v$. Moreover, in the limit $m_C/m_B \gg 1$, many collisions happen, but the net effect is like a single elastic collision between m_A and m_C mediated by box B, which has a maximum possible final speed of $2v$. Thus, the optimal setup is indeed $m_C \ll m_B \ll m_A$.

25. If a certain radioactive decay process happens n times per hour on average, then in any given hour, one expects to observe n decays with an uncertainty of $\sqrt{n}$. Each hour can be assumed independent of the previous one, and n can be assumed constant over time. How many hours do you need to conduct the observation so that you can determine n within an uncertainty of 1%?

(A) $n/10^2$

(B) $n/10^4$

(C) $10^2/n$

(D) $10^4/n$ $\leftarrow$ **CORRECT**

(E) $10^8/n^2$

Solution

Let the number of hours we observe be h. Then the number of decays observed will be, on average, hn. The standard deviation in the number of decays observed in an hour is $\sqrt{n}$. Each hour is independent of the ones before it. When we add h independent samples of a random variable, the sum has standard deviation of $\sqrt{h}\sigma$, with σ the standard deviation of the underlying distribution. So in this case, we will observe nh decays and the standard deviation in the number of decays observed is $\sqrt{hn}$. Our estimate for n will be

$$\frac{nh \pm \sqrt{nh}}{h} = n \pm \sqrt{\frac{n}{h}}.$$

We want the standard deviation to be about 1% of the correct value, so

$$\sqrt{\frac{n}{h}} \approx 0.01n.$$

Solving for nh, the total number of decays, we find

$$nh \approx 10^4.$$