A DIAGONAL METRIC WORKSHEET

Consider the following diagonal metric:

$$ds^2 = A(dx^1)^2 + B(dx^2)^2 + C(dx^3)^2 + D(dx^4)^2$$

In this metric, dx^1, dx^2, dx^3, dx^4 are completely arbitrary coordinates and *A, B, C,* and *D* are arbitrary functions of any or all of the coordinates. Note that:

1. Almost any metric of interest in GR can be cast into this form.
2. Selfless mathematicians have already tabulated $\Gamma^\alpha_{\mu\nu}$ and $R_{\mu\nu}$ for this metric (I am getting the results from Rindler, *Essential Relativity,* 2/e, Springer-Verlag, 1977.)
3. Because the metric components are all symmetrical, the formulas for $\Gamma^\alpha_{\mu\nu}$ and $R_{\mu\nu}$ also have a high degree of symmetry, which makes it easier to spot errors.
4. It can even be adapted for 3 (or 2) dimensional cases by setting *D* (or *C* and *D)* to 1 in the expressions below and treating the other coefficients as independent of x^4 (or x^4 and x^3).

In the expressions below, I will use the following shorthand notation:

$$\overline{A} \equiv \frac{1}{2A},\ \overline{B} \equiv \frac{1}{2B},\ \overline{C} \equiv \frac{1}{2C},\ \overline{D} = \frac{1}{2D};\quad A_1 \equiv \frac{\partial A}{\partial x^1},\ A_{12} \equiv \frac{\partial^2 A}{\partial x^1 \partial x^2},\ \text{etc.}$$

To use this worksheet, determine what *A, B, C,* and *D* are for your particular metric of interest, and write the results for that metric in the white space provided above each term.

CHRISTOFFEL SYMBOLS

$\Gamma^1_{11} = \overline{A}A_1,\qquad \Gamma^1_{12} = \Gamma^1_{21} = \overline{A}A_2,\qquad \Gamma^1_{13} = \Gamma^1_{31} = \overline{A}A_3,\qquad \Gamma^1_{14} = \Gamma^1_{41} = \overline{A}A_4$

$\Gamma^1_{22} = -\overline{A}B_1,\qquad \Gamma^1_{33} = -\overline{A}C_1,\qquad \Gamma^1_{44} = -\overline{A}D_1,\qquad$ others = 0

$\Gamma^2_{21} = \Gamma^2_{12} = \overline{B}B_1,\qquad \Gamma^2_{22} = \overline{B}B_2,\qquad \Gamma^2_{23} = \Gamma^2_{32} = \overline{B}B_3,\qquad \Gamma^2_{24} = \Gamma^2_{42} = \overline{B}B_4$

$\Gamma^2_{11} = -\overline{B}A_2,\qquad \Gamma^2_{33} = -\overline{B}C_2,\qquad \Gamma^2_{44} = -\overline{B}D_2,\qquad$ others = 0

$\Gamma^3_{31} = \Gamma^3_{13} = \overline{C}C_1,\qquad \Gamma^3_{32} = \Gamma^3_{23} = \overline{C}C_2,\qquad \Gamma^3_{33} = \overline{C}C_3,\qquad \Gamma^3_{34} = \Gamma^3_{43} = \overline{C}C_4$

$\Gamma^3_{11} = -\overline{C}A_3,\qquad \Gamma^3_{22} = -\overline{C}B_3,\qquad \Gamma^3_{44} = -\overline{C}D_3,\qquad$ others = 0

$\Gamma^4_{41} = \Gamma^4_{14} = \overline{D}D_1,\qquad \Gamma^4_{42} = \Gamma^4_{24} = \overline{D}D_2,\qquad \Gamma^4_{43} = \Gamma^4_{34} = \overline{D}D_3,\qquad \Gamma^4_{44} = \overline{D}D_4$

$\Gamma^4_{11} = -\overline{D}A_4,\qquad \Gamma^4_{22} = -\overline{D}B_4,\qquad \Gamma^4_{33} = -\overline{D}C_3,\qquad$ others = 0

RICCI TENSOR (sign convention: $R_{\mu\nu} = +R^{\alpha}{}_{\mu\alpha\nu}$)

$$R_{11} = 0 \quad -\bar{B}A_{22} \quad + \quad -\bar{C}A_{33} \quad + \quad -\bar{D}A_{44}$$

$$+ \quad 0 \quad -\bar{B}B_{11} \quad + \quad -\bar{C}C_{11} \quad + \quad -\bar{D}D_{11}$$

$$+ \quad 0 \quad \bar{B}^2 B_1^2 \quad + \quad \bar{C}^2 C_1^2 \quad + \quad \bar{D}^2 D_1^2$$

$$+ \quad 0 \quad + \quad \bar{A}A_1\bar{B}B_1 \quad + \quad \bar{A}A_1\bar{C}C_1 \quad + \quad \bar{A}A_1\bar{D}D_1$$

$$+ \quad \bar{B}A_2\bar{A}A_2 \quad + \quad \bar{B}A_2\bar{B}B_2 \quad + \quad -\bar{B}A_2\bar{C}C_2 \quad + \quad -\bar{B}A_2\bar{D}D_2$$

$$+ \quad \bar{C}A_3\bar{A}A_3 \quad + \quad -\bar{C}A_3\bar{B}B_3 \quad + \quad \bar{C}A_3\bar{C}C_3 \quad + \quad -\bar{C}A_3\bar{D}D_3$$

$$+ \quad \bar{D}A_4\bar{A}A_4 \quad + \quad -\bar{D}A_4\bar{B}B_4 \quad + \quad -\bar{D}A_4\bar{C}C_4 \quad + \quad +\bar{D}A_4\bar{D}D_4$$

$$R_{22} = -\bar{A}B_{11} \quad + \quad 0 \quad + \quad -\bar{C}B_{33} \quad + \quad -\bar{D}B_{44}$$

$$+ \quad -\bar{A}A_{22} \quad + \quad 0 \quad + \quad -\bar{C}C_{22} \quad + \quad -\bar{D}D_{22}$$

$$+ \quad \bar{A}^2 A_2^2 \quad + \quad 0 \quad + \quad \bar{C}^2 C_2^2 \quad + \quad \bar{D}^2 D_2^2$$

$$+ \quad \bar{A}B_1\bar{A}A_1 \quad + \quad \bar{A}B_1\bar{B}B_1 \quad + \quad -\bar{A}B_1\bar{C}C_1 \quad + \quad -\bar{A}B_1\bar{D}D_1$$

$$+ \quad \bar{B}B_2\bar{A}A_2 \quad + \quad 0 \quad + \quad \bar{B}B_2\bar{C}C_2 \quad + \quad \bar{B}B_2\bar{D}D_2$$

$$+ \quad -\bar{C}B_3\bar{A}A_3 \quad + \quad \bar{C}B_3\bar{B}B_3 \quad + \quad \bar{C}B_3\bar{C}C_3 \quad + \quad -\bar{C}B_3\bar{D}D_3$$

$$+ \quad -\bar{D}B_4\bar{A}A_4 \quad + \quad \bar{D}B_4\bar{B}B_4 \quad + \quad -\bar{D}B_4\bar{C}C_4 \quad + \quad \bar{D}B_4\bar{D}D_4$$

$$
\begin{aligned}
R_{33} = \; & -\overline{A}C_{11} && + && -\overline{B}C_{22} && + && 0 && + && -\overline{D}C_{44} \\
+ \; & -\overline{A}A_{33} && + && -\overline{B}B_{33} && + && 0 && + && -\overline{D}D_{33} \\
+ \; & \overline{A}^2 A_3^2 && + && \overline{B}^2 B_3^2 && + && 0 && + && \overline{D}^2 D_3^2 \\
+ \; & \overline{A}C_1\overline{A}A_1 && + && -\overline{A}C_1\overline{B}B_1 && + && \overline{A}C_1\overline{C}C_1 && + && -\overline{A}C_1\overline{D}D_1 \\
+ \; & -\overline{B}C_2\overline{A}A_2 && + && \overline{B}C_2\overline{B}B_2 && + && \overline{B}C_2\overline{C}C_2 && + && -\overline{B}C_2\overline{D}D_2 \\
+ \; & \overline{C}C_3\overline{A}A_3 && + && \overline{C}C_3\overline{B}B_3 && + && 0 && + && \overline{C}C_3\overline{D}D_3 \\
+ \; & -\overline{D}C_4\overline{A}A_4 && + && -\overline{D}C_4\overline{B}B_4 && + && \overline{D}C_4\overline{C}C_4 && + && \overline{D}C_4\overline{D}D_4
\end{aligned}
$$

$$
\begin{aligned}
R_{44} = \; & -\overline{A}D_{11} && + && -\overline{B}D_{22} && + && -\overline{C}D_{33} \\
+ \; & -\overline{A}A_{44} && + && -\overline{B}B_{44} && + && -\overline{C}C_{44} \\
+ \; & \overline{A}^2 A_4^2 && + && \overline{B}^2 B_4^2 && + && \overline{C}^2 C_4^2 \\
+ \; & \overline{A}D_1\overline{A}A_1 && + && -\overline{A}D_1\overline{B}B_1 && + && -\overline{A}D_1\overline{C}C_1 && + && +\overline{A}D_1\overline{D}D_1 \\
+ \; & -\overline{B}D_2\overline{A}A_2 && + && \overline{B}D_2\overline{B}B_2 && + && -\overline{B}D_2\overline{C}C_2 && + && \overline{B}D_2\overline{D}D_2 \\
+ \; & -\overline{C}D_3\overline{A}A_3 && + && -\overline{C}D_3\overline{B}B_3 && + && \overline{C}D_3\overline{C}C_3 && + && \overline{C}D_3\overline{D}D_3 \\
+ \; & \overline{D}D_4\overline{A}A_4 && + && \overline{D}D_4\overline{B}B_4 && + && \overline{D}D_4\overline{C}C_4
\end{aligned}
$$

$$R_{12} = -\bar{C}C_{12} + -\bar{D}D_{12} + \bar{C}^2 C_1 C_2 + \bar{D}^2 D_1 D_2$$
$$+ \bar{A}\bar{C}A_2 C_1 + \bar{A}\bar{D}A_2 D_1 + \bar{B}\bar{C}B_1 C_2 + \bar{B}\bar{D}B_1 D_2$$

$$R_{13} = -\bar{B}B_{13} + -\bar{D}D_{13} + \bar{B}^2 B_1 B_3 + \bar{D}^2 D_1 D_3$$
$$+ \bar{A}\bar{B}A_3 B_1 + \bar{A}\bar{D}A_3 D_1 + \bar{C}\bar{B}C_1 B_3 + \bar{C}\bar{D}C_1 D_3$$

$$R_{14} = -\bar{B}B_{14} + -\bar{C}C_{14} + \bar{B}^2 B_1 B_4 + \bar{C}^2 C_1 C_4$$
$$+ \bar{A}\bar{B}A_4 B_1 + \bar{A}\bar{C}A_4 C_1 + \bar{D}\bar{B}D_1 B_4 + \bar{D}CD_1 C_4$$

$$R_{23} = -\bar{A}A_{23} + -\bar{D}D_{23} + \bar{A}^2 A_2 A_3 + \bar{D}^2 D_2 D_3$$
$$+ \bar{B}\bar{A}B_3 A_2 + \bar{B}\bar{D}B_3 D_2 + \bar{C}\bar{A}C_2 A_3 + \bar{C}\bar{D}C_2 D_3$$

$$R_{24} = -\bar{A}A_{24} + -\bar{C}C_{24} + \bar{A}^2 A_2 A_4 + \bar{C}^2 C_2 C_4$$
$$+ \bar{B}\bar{A}B_4 A_2 + \bar{B}\bar{C}B_4 C_2 + \bar{D}\bar{A}D_2 A_4 + \bar{D}\bar{C}D_2 C_4$$

$$R_{34} = -\bar{A}A_{34} + -\bar{B}B_{34} + \bar{A}^2 A_3 A_4 + \bar{B}^2 B_3 B_4$$
$$+ \bar{C}\bar{A}C_4 A_3 + \bar{C}\bar{B}C_4 B_3 + \bar{D}\bar{A}D_3 A_4 + \bar{D}\bar{B}D_3 B_4$$