Teaching General Relativity—A Seven-Layer Cake

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(Dated: July 18, 2006)

Abstract

General Relativity is now recognized as central to some of the most dynamic fields in science, including cosmology, particle physics, and gravitational-wave astronomy. Its key ideas can be taught at all levels. This can be divided into seven "layers": noncalculus introductory courses; calculus-based introductory courses; Modern Physics courses; specialized undergraduate courses; undergraduate research; graduate courses; and graduate research. In a high-school course the principle of relativity, spacetime geometry, and connections between gravitation and geometry can be introduced via suitable illustrations. In a calculus-based course these can be supplemented by calculations in special-relativistic mechanics. In Modern Physics calculations involving gravitation and spacetime metrics can be introduced. A specialized undergraduate course can include geodesics and mechanics in curved spacetime, tensors, and the Einstein field equations. Undergraduates' research can involve quite sophisticated theory, provided they are given suitably limited problems.

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I. INTRODUCTION

For many decades after its introduction, the General Theory of Relativity was relegated to the margins of physics, considered by many to be all but irrelevant except, perhaps, in the inaccessible early universe. Even most Ph. D.'s in physics never studied it. That will no longer do: Now, the theory is a foundation of some of the most dynamic 21st-century science, including cosmology, elementary-particle physics, and gravitational-wave astronomy. It is therefore appropriate and necessary that its key ideas be incorporated into the teaching of physics at all levels, not only in the graduate course or the doctoral dissertation.

The teaching of general relativity can be divided into seven stages or "layers": the high-school or algebra/trig-based introductory course; the calculus-based introductory course; the Modern Physics course; the specialized, upper-division undergraduate course; the undergraduate research project; the graduate course; and graduate research work. The layers are distinguished by the ideas which can profitably be introduced and the results and techniques which can be explored there. The time available and the background of the the students determine the character of the material appropriate to each layer.

In this article I shall describe briefly my efforts and experiences introducing general relativity into physics teaching in the first five layers, in classes, presentations, and undergraduate-research supervision at Saint Louis University over the last thirteen years. In each layer I strive to emphasize the ideas of special and general relativity as logical, even necessary outgrowths of the physics which has gone before. Thus I de-emphasize the bizarre, counter-intuitive, and speculative—kinematical "paradoxes," wormholes and time travel, p-branes in higher dimensions. After all, if we emphasize sufficiently to students that contemporary physics is bizarre and speculative, they will (correctly) infer that there is no reason to learn it. As I am a theorist, I tend to give ideas and calculations precedence over detailed descriptions of experiments.

II. IN THE NONCALCULUS INTRODUCTORY COURSE

In a typical two-semester algebra/trig-based introductory course, the Special Theory of Relativity can be introduced in the mechanics portion, after Newton's Laws, energy, and momentum are studied. I emphasize the *motivation* for the theory—to resolve the inconsistency between Newton's Laws, with their Galilean invariance, and the Maxwell equations of electromagnetism. This leads immediately to the Lorentz transformations. I do not present relativistic kinematics in great detail—there isn't time for it—but proceed directly to Einstein's redefinition of energy and momentum, leading to the description of non-Newtonian processes in which mass and kinetic energy are interchanged. The Lorentz transformations imply the concept of spacetime geometry, allowing me to discuss the General Theory of Relativity subsequently. All of this description, of course, requires no calculus.

I introduce the foundations of the General Theory of Relativity following the treatment of gravitation and orbits. I begin by identifying what is general about the general theory: In order to expand mechanics to encompass all reference frames, inertial forces must be included. Such a theory must include gravity, because gravity *is* an inertial force: It is proportional to the mass of the object effected—making trivial the equality of inertial and gravitational mass—and it can be eliminated by a change of reference frame. The latter can be vividly illustrated both via the familiar Einstein elevator, and via a (purely Newtonian) treatment of one projectile seen from another—the classic monkey-and-hunter problem, in the service of the Principle of Equivalence! The inclusion of gravity demands curved spacetime geometry. This too can be vividly illustrated without any differential geometry. I rely on two illustrations: First, I identify the tidal distortion of an array of masses in an Einstein elevator with the convergence of meridians on the surface of the Earth. Second, I consider the four-velocities of a dense object falling through an idealized Earth and returning, and a similar object thrown upward so as to fall back in the same time. The change in angle between these vectors is a signature of curvature, exactly as is the rotation of an arrow carried along a spherical triangle on the Earth's surface. With curvature introduced, I can briefly and qualitatively describe the Einstein field equations and the most important consequences of the theory: measurable effects on orbits and signals, black holes, gravitational waves, and cosmology. Even at this level, I use Kruskal diagrams to illustrate the nature of black holes; I have seen dreadfully misleading depictions of black holes even in very recent texts.

III. IN THE CALCULUS-BASED INTRODUCTORY COURSE

The fundamental ideas of special and general relativity can be introduced in the same way in a calculus-based introductory course. (Most texts now include a chapter on relativity in the “modern” section toward the end, but I find it in that position impossible to reach in a two-semester, three- or four-credit-hour course.) Here the students’ more extensive mathematics background makes possible more detailed explorations and calculations. I introduce proper time, four-vectors, and four-velocities, in order to motivate the relativistic definitions of energy and momentum. And I make extensive use of the relativistic rocket to allow the students to examine relativistic dynamics in detail; the integrations required are straightforward at this level.

My treatment of general relativity in this course is similar to that in the noncalculus introductory course. Time constraints generally do not allow me to introduce the mathematics—spacetime metrics, for example—needed to go into more detail.

IV. IN THE MODERN PHYSICS COURSE

A one-semester, three-credit-hour Modern Physics course, such as we use at Saint Louis University, is usually based largely on phenomenology. It typically contains only a brief introduction to relativity; cosmology, in recent texts, is treated as an offshoot of particle physics. I teach the course from a more theoretical perspective, however, and divide it into three blocks of roughly equal duration: Special relativity, general relativity, and cosmology; quantum mechanics and its applications; quantum field theory and elementary-particle physics. By the time students take this course, they have completed the calculus sequence and are taking or have taken an introduction to differential equations. With more time and a stronger mathematical foundation, the exploration of relativity in this course can be much more extensive and detailed than that in introductory courses.

At this level I explore special-relativistic kinematics in detail, deriving kinematical effects, resolving “paradoxes,” introducing spacetime diagrams and the spacetime metric. The students and I solve the “constant acceleration” problem and examine this motion analytically, numerically, and graphically. Our study of dynamics includes four-momentum, the relativistic rocket, and scattering problems.

With the notion of spacetime metric already introduced, the students can undertake actual calculations in general relativity, to supplement the qualitative introduction of the lower-level courses. They see the Newtonian metric and the Schwarzschild metric, and carry out simple gravitational-redshift/time-dilation calculations. We use the Kruskal diagram to examine the nature of black holes. I describe the Einstein field equations qualitatively, and briefly discuss gravitational waves and gravitational-wave astronomy.

In the cosmology portion of the course, I show the students the Friedmann-Robertson-Walker metric, and the connection between the evolution of the Universe and its contents. I review briefly the chief features of the standard hot-Big-Bang model, the evidence supporting it, and the most recent cosmological discoveries. Naturally this last material is updated every time I teach the course.

Our Modern Physics course has a companion laboratory course. Special-relativistic phenomena are illustrated in some of the experiments, such as gamma-ray spectroscopy and muon-lifetime measurements. Suitable experiments involving general relativity, e.g., gravitational-redshift measurement (a compact version of the classic Pound-Rebka experiment) are not presently available.

V. AN UNDERGRADUATE GENERAL RELATIVITY COURSE

This is the highest layer of undergraduate classroom relativity instruction. I have twice been allowed to teach a one-semester, three-credit-hour, senior-level special topics course in general relativity. Both times this was done at the request of a small group of physics and engineering students, almost all of whom had taken Modern Physics and advanced mathematics through differential equations and linear algebra. Naturally I consider this course an ongoing experiment, still in development.

Once extremely rare, undergraduate general relativity courses are now being developed at many schools, using a variety of approaches—hence the present workshop. My own experience leads me to approach the subject conceptually, introducing the mathematical tools necessary to treat the basic framework of the theory and its principal results—as opposed, for example, to an approach leaning more heavily on experiments and phenomenology. A mathematician colleague has suggested introducing a one-semester undergraduate differential geometry course as preparation for this course. This, or perhaps a second semester of the relativity course, remain possibilities for future development, depending on the success of the current version of the course.

I used as texts for the course *Spacetime Physics,* by E. F. Taylor and J. A. Wheeler (Freeman, New York, 1992) and *Gravitation,* by C. W. Misner, K. S. Thorne, and J. A. Wheeler

(Freeman, New York, 1973)—well known as "MTW"—the first time I taught it. The second time, I used the new text *Gravity: An Introduction to Einstein's General Relativity,* by J. B. Hartle (Addison-Wesley, San Francisco, 2003) and again MTW. The use of MTW in an undergraduate course may seem surprising, as it is a famous graduate text, but I include it because its coverage is encyclopedic, even if it does not contain the most current results; its introductory ("Track One") sections are accessible to advanced undergraduates; and I believe there is such a thing as classic literature in science, which all students should read.

My syllabus for the course reads thus:

- Special Relativity
  - Kinematics and Dynamics
  - Spacetime Geometry
  - Tensors and Why We Love Them
  - Relativity and Electrodynamics
- General Relativity
  - Geodesics in Curved Spacetime
  - Planets, Stars, and Black Holes
  - The Einstein Field Equations
  - Gravitational Waves
- Cosmology
  - Cosmological Spacetime Geometries
  - Why the Big Bang is a Lock
  - Inflation
  - Cosmology in Renaissance

The treatment of special relativity reviews material from the Modern Physics course (which many of the students have seen there). I then introduce tensors, beginning with their definitions both as quantities with particular transformation properties, and as linear maps between vector spaces. The Faraday field tensor and the stress-energy tensor provide concrete examples. This allows me to show the Lorentz invariance of the Maxwell equations, and the intimate connection between relativity and electrodynamics, quickly and elegantly. The chief difficulties I have encountered with this portion of the course are students' unfamiliarity with matrix manipulations in symbolic, component form, and the hurdle of the Einstein summation convention.

After reviewing the motivation for the introduction of curved spacetime geometry described qualitatively in the lower-level courses, I use geodesic Lagrangian methods to introduce connections, covariant derivatives, and geodesic equations. This provides an opportunity to show the students basic variational methods, which are not always strongly emphasized in our undergraduate classical mechanics course. We examine the Newtonian-limit metric and the concordance between Einsteinian and Newtonian gravitation in that limit. I conclude this section with a brief description of the Riemann, Ricci, and Weyl curvature tensors, for later use.

Orbital mechanics in the Schwarzschild geometry forms the basis of my treatment both of Solar-System gravitation and black-hole physics. This allows the students to practice the general techniques they have learned and to see differences between strong and weak gravity. We examine perturbed orbits in this geometry, deriving classic results such as perihelion precession and, incidentally, introducing the students to perturbative methods not usually seen in lower-level physics courses.

I then move to the "active" side of the theory, briefly examining the Einstein field equation, its complexities, and its uses. This leads into a discussion of gravitational radiation, including very basic descriptions of wave solutions and current efforts in gravitational-wave astronomy. This portion of the course tends to be brief and cursory, due to time constraints.

The final portion of the course is devoted to cosmology, as a remarkable influx of new data has made this one of the most dynamic fields in contemporary science. I emphasize the foundations of the subject more strongly than the latest discoveries (which may be radically revised with new experiments), beginning with Friedmann-Robertson-Walker geometries and the observations which make Big-Bang cosmology "the standard model." I include descriptions of inflation theory and experimental results of the last decade as the course concludes.

This version of an undergraduate general relativity course has proved reasonably successful, based on the responses of students who have completed it. (Of course, the students are favorably disposed toward the subject by self-selection.) I shall continue to develop and modify this approach—each class taught is a learning experience.

VI. UNDERGRADUATE RESEARCH IN GENERAL RELATIVITY

Often undergraduate research is experimental in nature, as experimenters can use "extra hands" in ways theorists do not. But I have found that undergraduates can do independent research in general relativity theory—in which they both accomplish something significant and appreciate what they are doing—provided that they acquire the appropriate background knowledge and pursue suitably tailored research questions.

At Saint Louis University all of our Bachelor of Science candidates in physics are required to carry out a three-semester undergraduate research program, usually ending with an oral presentation to the department (and sometimes with a conference presentation or published paper). Students choose the area of their research by consulting with individual faculty members in their fourth or fifth semester of study. To date I have supervised five students in their research, including one just starting; four of these have tackled problems in general relativity and cosmology. Three of the five have now completed their programs and gone on to graduate studies at distinguished institutions.

My approach is to have the students devote the first semester, and usually some time in the intervening summer break, to acquiring the necessary background. In the cases of the students working in relativity and cosmology, this means material at the level covered in the Modern Physics course and the relativity course described above (which, in most cases, they have taken), plus additional material as needed. The students read and work through suitable texts, meeting regularly with me to discuss and review. They then apply this to a research problem—often, an offshoot of my own research—which I have carefully pruned so that the student is exploring something novel, but can complete at least a significant component of it in the remaining two semesters. For example, I have had one student calculate and display via computer animation galaxy redshift distributions in a cell-lattice cosmological model, elegantly fusing relativity, geometry, basic astronomy, and computer graphics; another calculated tree-level quantum-electrodynamic scattering in de Sitter space; another is developing a classification scheme for Friedmann-Robertson-Walker universes with all possible light and dark matter and energy contents.

Each new student represents another learning experience in developing my undergraduate research program in general relativity theory. Each time I gain greater understanding of what students need, where they encounter difficulties, what they can do. But the results I have seen so far are very encouraging indeed.

VII. GRADUATE COURSES AND DISSERTATION RESEARCH

The final layers of the general-relativity cake are the courses offered at the graduate level and masters' theses and doctoral dissertations in the field. Since Saint Louis University does not at present have graduate degree programs in physics, my experience with these layers is limited to my own and my peers'—as students. But it is quite clear that below these layers, beginning even at the high school level, in the introductory courses, and in upper-division undergraduate courses and independent research, the inclusion of the ideas and consequences of general relativity is possible, productive, and long overdue.