HYPERBOLA GEOMETRY

Euclidean distance is based on the unit circle; the set of points which are unit distance from the origin. Hyperbolic geometry is obtained simply by using a different distance function! Measure the “separation distance” of a point \( B = (x, y) \) from the origin using the definition:

\[ d_{H}(O, B) = \sqrt{x^2 - y^2} \]

Then, as shown below the unit “circle” becomes the unit hyperbola

\[ x^2 - y^2 = 1 \]

and we further restrict ourselves to the branch with \( x > 0 \) if \( B \) is a point on this hyperbola, then we can define the hyperbolic angle \( \beta \) between the line from the origin to \( B \) and the (positive) \( x \)-axis to be the Lorentzian length of the arc of the unit hyperbola between \( B \) and the point \((1, 0)\), where \( d_{H}(O, (1, 0)) = \sqrt{2} \). We could then define the hyperbolic trig functions to be the coordinates \((x, y)\) of \( B \), that is

\[
\begin{align*}
\cosh \beta &= x \\
\sinh \beta &= y
\end{align*}
\]

and by symmetry, the point \( A \) on this hyperbola has coordinates \((x, y) = (\sinh \beta, \cosh \beta)\). See the figure below. A little work shows that this definition is exactly the same as the standard one, namely

\[
\begin{align*}
\cosh \beta &= \frac{e^\beta + e^{-\beta}}{2} \\
\sinh \beta &= \frac{e^\beta - e^{-\beta}}{2}
\end{align*}
\]

To see this, use \( x^2 - y^2 = 1 \) to compute

\[ dH(O, (x, y)) = \sqrt{x^2 - y^2} \]

then take the square root of either expression and integrate. (The integrals are hard.) Finally, solve for \( x \) or \( y \) in terms of \( \beta \).

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REFERENCES


TRIANGLE TRIG

We now recast ordinary triangle trig into hyperbola geometry. Suppose you know \( \tan \beta = \frac{y}{x} \) and you wish to determine \( \cosh \beta \).

One can of course do this algebraically, using the identity

\[ \cosh \beta = \frac{1}{\sqrt{1 - \sinh^2 \beta}} \]

But it is easier to draw an equilateral triangle containing an angle whose hyperbolic tangent is \( \frac{y}{x} \). In this case, the obvious choice would be the triangle shown above, with sides \( 4 \) and 5.

What is \( \cosh \beta \)? Well, we first need to work out the length \( \delta \) of the hypotenuse. The (hyperbolic) Pythagorean Theorem tells us that

\[ x^2 - y^2 = \delta^2 \]

so \( \delta = 4 \). Take a good look at this 3-4-5 triangle of hyperbola geometry. Note that we know all the sides of the triangle, it is easy to see that \( \cosh \beta = \frac{5}{4} \).

LORENTZ TRANSFORMATIONS

The Lorentz transformation from a moving frame \((x', y')\) to a frame \((x, y)\) at rest is given by

\[
\begin{align*}
x &= \gamma (x' + v y') \\
y &= y'
\end{align*}
\]

where \( \gamma = \frac{1}{\sqrt{1 - v^2}} \). We can simplify things still further: Introduce the rapidity \( \beta \) via

\[ \gamma = \cosh \beta \]

Inserting this into the expression for \( \gamma \) we obtain

\[
\gamma = \sqrt{1 + \tanh^2 \beta} = \sqrt{1 + \sinh^2 \beta} = \frac{1}{\cos \beta}
\]

and

\[ \tanh \beta = \frac{\sinh \beta}{\cosh \beta} = \frac{y}{x} \]

Inserting these identities into the Lorentz transformations above brings them to the remarkably simple form

\[ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cosh \beta & \sinh \beta \\ \sinh \beta & \cosh \beta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \]

Lorentz transformations are just hyperbolic rotations!