AAPT UNITED STATES PHYSICS TEAM
AIP 2014

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination consists of two parts.
- Part A has four questions and is allowed 90 minutes.
- Part B has two questions and is allowed 90 minutes.
- The first page that follows is a cover sheet. Examinees may keep the cover sheet for both parts of the exam.
- The parts are then identified by the center header on each page. Examinees are only allowed to do one part at a time, and may not work on other parts, even if they have time remaining.
- Allow 90 minutes to complete Part A. Do not let students look at Part B. Collect the answers to Part A before allowing the examinee to begin Part B. Examinees are allowed a 10 to 15 minutes break between parts A and B.
- Allow 90 minutes to complete Part B. Do not let students go back to Part A.
- Ideally the test supervisor will divide the question paper into 4 parts: the cover sheet (page 2), Part A (pages 3-16), Part B (pages 17-23), and several answer sheets for two of the questions in part A (pages 25-28). Examinees should be provided parts A and B individually, although they may keep the cover sheet. The answer sheets should be printed single sided!
- The supervisor *must* collect all examination questions, including the cover sheet, at the end of the exam, as well as any scratch paper used by the examinees. Examinees may *not* take the exam questions. The examination questions may be returned to the students after April 15, 2014.
- Examinees are allowed calculators, but they may not use symbolic math, programming, or graphic features of these calculators. Calculators may not be shared and their memory must be cleared of data and programs. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. Examinees may not use any tables, books, or collections of formulas.

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INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Work Part A first. You have 90 minutes to complete all four problems. Each question is worth 25 points. Do not look at Part B during this time.
- After you have completed Part A you may take a break.
- Then work Part B. You have 90 minutes to complete both problems. Each question is worth 50 points. Do not look at Part A during this time.
- Show all your work. Partial credit will be given. Do not write on the back of any page. Do not write anything that you wish graded on the question sheets.
- Start each question on a new sheet of paper. Put your AAPT ID number, your name, the question number and the page number/total pages for this problem, in the upper right hand corner of each page. For example,

  AAPT ID #
  Doe, Jamie
  A1 - 1/3

- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- Questions with the same point value are not necessarily of the same difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers/solutions) on this contest until after April 15, 2014.**

Possibly Useful Information. You may use this sheet for both parts of the exam.

$g = 9.8\ \mathrm{N/kg}$	$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}\ \mathrm{T \cdot m/A}$
$c = 3.00 \times 10^8\ \mathrm{m/s}$	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J/K}$
$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol \cdot K)}$
$\sigma = 5.67 \times 10^{-8}\ \mathrm{J/(s \cdot m^2 \cdot K^4)}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$1\mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s} = 4.14 \times 10^{-15}\ \mathrm{eV \cdot s}$
$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x\rvert \ll 1$
$\sin\theta \approx \theta - \frac{1}{6}\theta^3$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for $\lvert\theta\rvert \ll 1$

Part A

Question A1

Inspired by: http://www.wired.com/wiredscience/2012/04/a-leaning-motorcycle-on-a-vertical-wall/

A unicyclist of total height h goes around a circular track of radius R while leaning inward at an angle θ to the vertical. The acceleration due to gravity is g.

a. Suppose $h \ll R$. What angular velocity ω must the unicyclist sustain?

Solution

Work in the rotating frame, where four forces act on the unicyclist: a normal and frictional force at the point of contact, gravity downwards at the center of mass, and a (fictitious) centrifugal force.

If $h \ll R$, all parts of the unicyclist are at a distance of approximately R from the center of the circle, so the centripetal acceleration of every part of the unicyclist is $\omega^2 R$. The centrifugal force can then be taken to act at the center of mass for purposes of computing the torque. If the center of mass is a distance l from the point of contact, the torque about the point of contact is

$$\tau = m\omega^2 Rl\cos\theta - mgl\sin\theta$$

Since the unicyclist is stationary in this frame, $\tau = 0$:

$$m\omega^2 Rl\cos\theta - mgl\sin\theta = 0$$

$$\omega = \sqrt{\frac{g}{R}\tan\theta}$$

b. Now model the unicyclist as a uniform rod of length h, where h is less than R but not negligible. This refined model introduces a correction to the previous result. What is the new expression for the angular velocity ω? Assume that the rod remains in the plane formed by the vertical and radial directions, and that R is measured from the center of the circle to the point of contact at the ground.

Solution

The centripetal acceleration now varies meaningfully along the length of the unicyclist. In the rotating frame, the torque about the point of contact is given by

$$\tau_c = \int \omega^2 rz\, dm$$

where r is the distance from the center of the circle, z is the height above the ground, and dm is a mass element. Because the mass of the unicyclist is uniformly distributed along a length h, the mass element dm can be written as $\frac{m}{h}\, ds$ for a length element ds, and we have

$$\tau_c = \int_0^h \omega^2 (R - s\sin\theta)(s\cos\theta)\, \frac{m}{h}\, ds$$

$$\tau_c = m\omega^2 h \cos\theta \left(\frac{R}{2} - \frac{h}{3}\sin\theta\right)$$

Gravity continues to act at the center of mass, a distance $\frac{h}{2}$ from the point of contact, and in the opposite direction:

$$\tau_g = -mg\frac{h}{2}\sin\theta$$

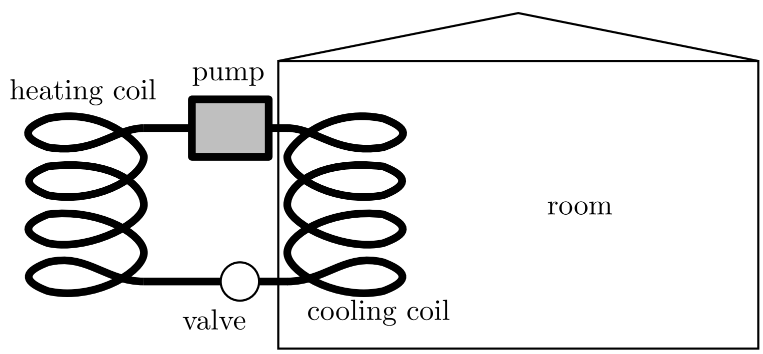
Again, the total torque is zero, so

$$m\omega^2 h \cos\theta \left(\frac{R}{2} - \frac{h}{3}\sin\theta\right) - mg\frac{h}{2}\sin\theta = 0$$

$$\omega = \sqrt{\left(\frac{g}{R}\tan\theta\right)\left(1 - \frac{2}{3}\frac{h}{R}\sin\theta\right)^{-1}}$$

Question A2

A room air conditioner is modeled as a heat engine run in reverse: an amount of heat Q_L is absorbed from the room at a temperature T_L into cooling coils containing a working gas; this gas is compressed adiabatically to a temperature T_H; the gas is compressed isothermally in a coil *outside* the house, giving off an amount of heat Q_H; the gas expands adiabatically back to a temperature T_L; and the cycle repeats. An amount of energy W is input into the system every cycle through an electric pump. This model describes the air conditioner with the best possible efficiency.



Assume that the outside air temperature is T_H and the inside air temperature is T_L. The air-conditioner unit consumes electric power P. Assume that the air is sufficiently dry so that no condensation of water occurs in the cooling coils of the air conditioner. Water boils at 373 K and freezes at 273 K at normal atmospheric pressure.

a. Derive an expression for the maximum rate at which heat is removed from the room in terms of the air temperatures T_H, T_L, and the power consumed by the air conditioner P. Your derivation must refer to the entropy changes that occur in a Carnot cycle in order to receive full marks for this part.

Solution

From Carnot cycles, and by entropy conservation, we have

$$\frac{Q_H}{Q_L} = \frac{T_H}{T_L}$$

Also, by energy conservation,

$$Q_H = Q_L + W$$

So heat is removed at a rate Q_L/t. But

$$Q_L = Q_H - W = Q_L \frac{T_H}{T_L} - W$$

or

$$W = Q_L \left(\frac{T_H}{T_L} - 1 \right)$$

Rearrange and divide by time,

$$\frac{Q_L}{t} = P \left(\frac{T_L}{T_H - T_L} \right)$$

b. The room is insulated, but heat still passes into the room at a rate $R = k\Delta T$, where ΔT is the temperature difference between the inside and the outside of the room and k is a constant. Find the coldest possible temperature of the room in terms of T_H, k, and P.

Solution

Equate.

$$k\Delta T = P \frac{T_L}{\Delta T} = P \frac{T_H - \Delta T}{\Delta T}$$

or

$$k(\Delta T)^2 = PT_H - P\Delta T,$$

which is a quadratic that can be solved as

$$\Delta T = \frac{-P \pm \sqrt{P^2 + 4PkT_H}}{2k},$$

but only the positive root has physical significance. Writing $x = P/k$,

$$\Delta T = \frac{x}{2} \left(\sqrt{1 + 4T_H/x} - 1 \right)$$

That's the amount the room is colder than the outside, so

$$T_L = T_H - \frac{x}{2} \left(\sqrt{1 + 4T_H/x} - 1 \right)$$

c. A typical room has a value of $k = 173$ W/°C. If the outside temperature is 40°C, what minimum power should the air conditioner have to get the inside temperature down to 25°C?

Solution

Don't forget to convert to Kelvin!

From above,

$$P = \frac{k(\Delta T)^2}{T_L},$$

so

$$P = 130 \text{ W}$$

Question A3

When studying problems in special relativity it is often the invariant distance Δs between two events that is most important, where Δs is defined by

$$(\Delta s)^2 = (c\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]$$

where $c = 3 \times 10^8$ m/s is the speed of light.[1]

a. Consider the motion of a projectile launched with initial speed v_0 at angle of θ_0 above the horizontal. Assume that g, the acceleration of free fall, is constant for the motion of the projectile.

   i. Derive an expression for the invariant distance of the projectile as a function of time t as measured from the launch, assuming that it is launched at $t = 0$. Express your answer as a function of any or all of θ_0, v_0, c, g, and t.

   ii. The radius of curvature of an object's trajectory can be estimated by assuming that the trajectory is part of a circle, determining the distance between the end points, and measuring the maximum height above the straight line that connects the endpoints. Assuming that we mean "invariant distance" as defined above, find the radius of curvature of the projectile's trajectory as a function of any or all of θ_0, v_0, c, and g. Assume that the projectile lands at the same level from which it was launched, and assume that the motion is *not* relativistic, so $v_0 \ll c$, and you can neglect terms with v/c compared to terms without.

Solution

The particle begins at $ct = x = y = 0$ and takes a path satisfying

$$x = v_0 t \cos\theta_0$$

$$z = v_0 t \sin\theta_0 - \frac{1}{2} g t^2$$

Thus

$$s^2 = (ct)^2 - (v_0 t \cos\theta_0)^2 - (v_0 t \sin\theta_0 - \frac{1}{2} g t^2)^2$$

which can be simplified to

$$s^2 = (c^2 - v_0^2)t^2 + \frac{1}{2} g v_0 \sin\theta_0 \, t^3 + \frac{1}{4} g^2 t^4$$

b. The particle reaches the ground again at

$$t_f = 2\frac{v_0 \sin\theta}{g}$$

$$x_f = v_0 \cos\theta t_f$$

$$z_f = 0$$

[1] We are using the convention used by Einstein

and so the invariant distance between the endpoints is

$$s^2 = (ct_f)^2 + (v_0 \cos\theta\, t_f)^2$$

$$s \approx 2c\frac{v_0 \sin\theta}{g}$$

The maximum height above the ground is

$$z_{max} = \frac{(v_0 \sin\theta)^2}{2g}$$

Because $z_{max} \ll s$, we have from similar triangles

$$\frac{z_{max}}{\frac{1}{2}s} \approx \frac{\frac{1}{2}s}{R}$$

$$R \approx \frac{1}{4}\frac{s^2}{z_{\text{max}}}$$

$$R \approx 2\frac{c^2}{g}$$

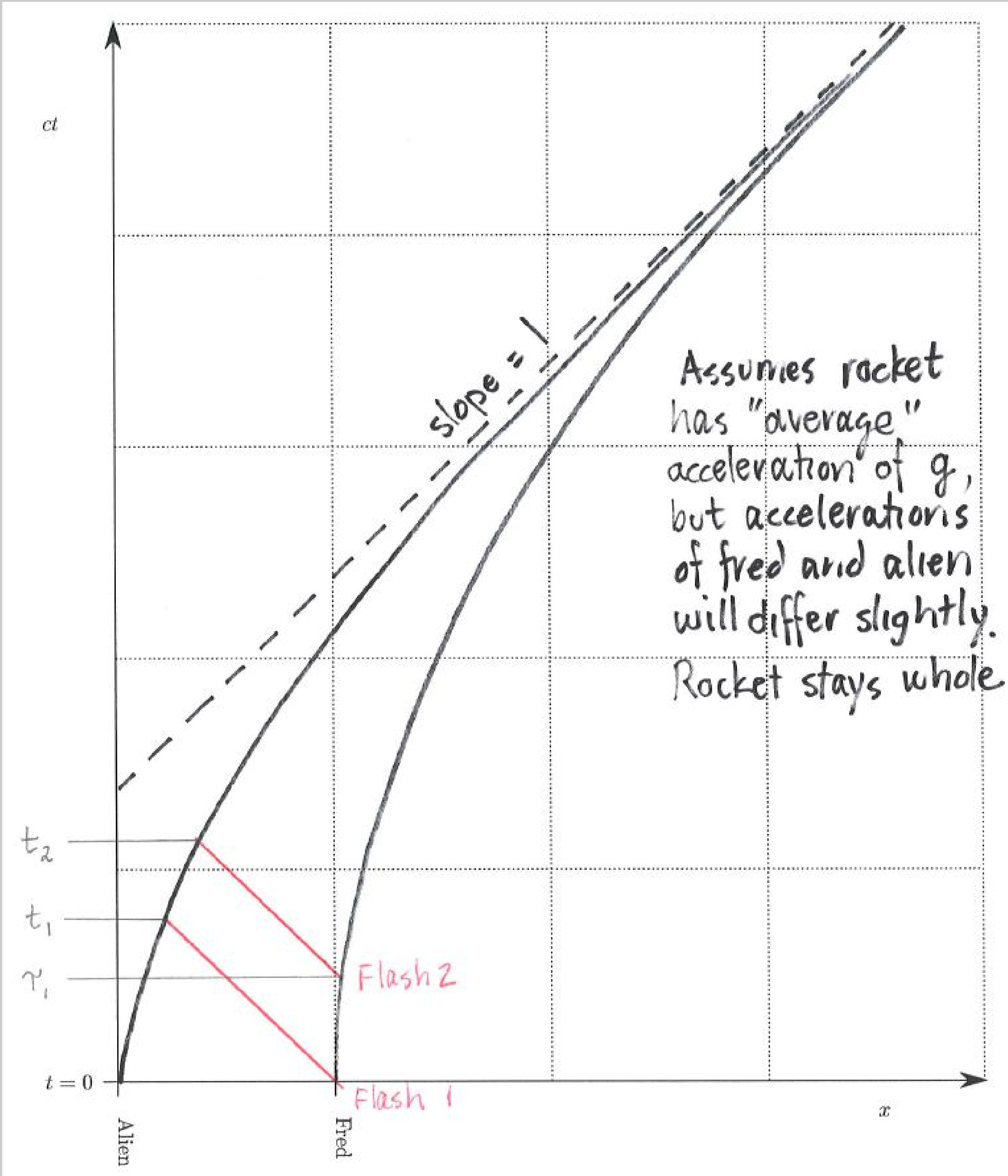
c. A rocket ship far from any gravitational mass is accelerating in the positive x direction at a constant rate g, as measured by someone *inside* the ship. Spaceman Fred at the right end of the rocket aims a laser pointer toward an alien at the left end of the rocket. The two are separated by a distance d such that $dg \ll c^2$; you can safely ignore terms of the form $(dg/c^2)^2$.

i. Sketch a graph of the motion of both Fred and the alien on the space-time diagram provided in the answer sheet. The graph is *not* meant to be drawn to scale. Note that t and x are reversed from a traditional graph. Assume that the rocket has velocity $v = 0$ at time $t = 0$ and is located at position $x = 0$. Clearly indicate any asymptotes, and the slopes of these asymptotes.

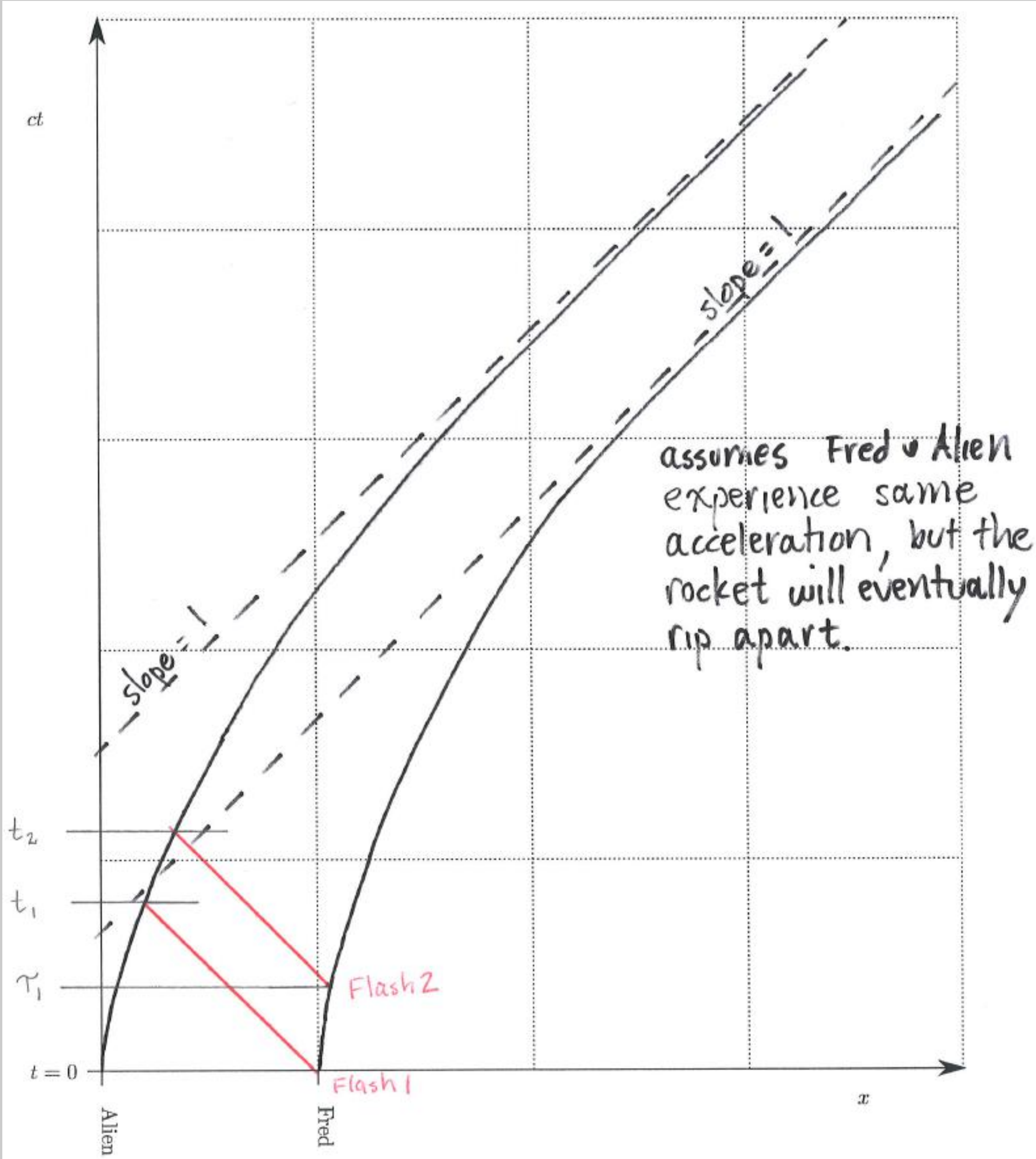
Solution

Since the rocket-ship can never exceed the speed of light, yet it is always accelerating (in the local frame), it must approach an asymptote that has a slope of one on the space-time diagram shown. There is a slight challenge to consider, however. Since the rocket ship is an extended object, do the two ends (represented by Fred and the Alien) approach the same asymptote, or two different asymptotes?

At this point we must remember a consequence of special relativity for a ship moving at relativistic speeds: the ship will contract in length as measured in the original frame. As the speed of the ship approaches that of light, the length of the ship will approach zero. The only way for that to happen is for the two ends of the ship to have slightly different accelerations.



If you had assumed (somewhat incorrectly) that the two ends of the ship have the same acceleration, then the two trajectories would be approaching two different asymptotes separated by a constant horizontal distance. But this would mean the apparent length of the ship was constant, regardless of speed. In the instantaneous rest frame of the ship we then require that Fred and the Alien be moving apart. This means that the ship must be stretching, and eventually breaking.



ii. If the frequency of the laser pointer as measured by Fred is f_1, determine the frequency of the laser pointer as observed by the alien. It is reasonable to assume that $f_1 \gg c/d$.

Solution

If the spaceship is uniformly accelerating then we can choose a reference frame which is instantaneously at rest with respect to the spaceship at $t = 0$.

Consider two instantaneous flashes from the astronaut. Flash 1 is emitted by Fred at $t = 0$, flash 2 is emitted by the Fred at $t = \tau_1$.

Flash 1 travels down towards the alien, who is accelerating upward. Let t_1 be the time at which the alien sees flash 1. Equate the distances,

$$ct_1 = d - \frac{1}{2}g{t_1}^2.$$

Flash 2 is emitted at τ_1. Flash 2 travels down towards the alien, who is still accelerating upward. Let t_2 be the time at which the alien sees flash 2. Equate the distances,

$$c(t_2 - \tau_1) = \frac{1}{2}g{\tau_1}^2 + d - \frac{1}{2}gt_2 2.$$

Notice that the pulse travel for a time $t_2 - \tau_1$! Defining $t_2 =_1 +\tau_2$, and then expanding to keep terms linear in $\tau \ll t_1$,

$$c(t_1 + \tau_2 - \tau_1) = h - \frac{1}{2}g{t_1}^2 - gt_1\tau_2.$$

Combining,

$$c(\tau_2 - \tau_1) = -gt_1\tau_2.$$

but $t_1 \approx d/c$, so

$$\tau_2 = \tau_1\left(1 - \frac{gh}{c^2}\right)$$

In terms of frequency this is

$$f_2 = f_1\left(1 + \frac{gd}{c^2}\right)$$

Alternatively, we can follow the motion of two wave crests from Fred to the alien. We can work in the reference frame where Fred is stationary when the first crest is emitted. Because $f_1 \gg c/d$, we can assume that Fred remains stationary during the period between the first and second crests; then, because the first crest moves towards the alien at a speed c, they are separated by a distance c/f_1.

Because $dg \ll c^2$, the time taken for the crests to reach the alien is due almost entirely to the motion of the crests, and is d/c. In this time, the spaceship accelerates to a speed gd/c, and (in Fred's frame of reference) the relative speed of the crests and the alien is $c + \frac{gd}{c}$. The time between crests reaching the alien is thus

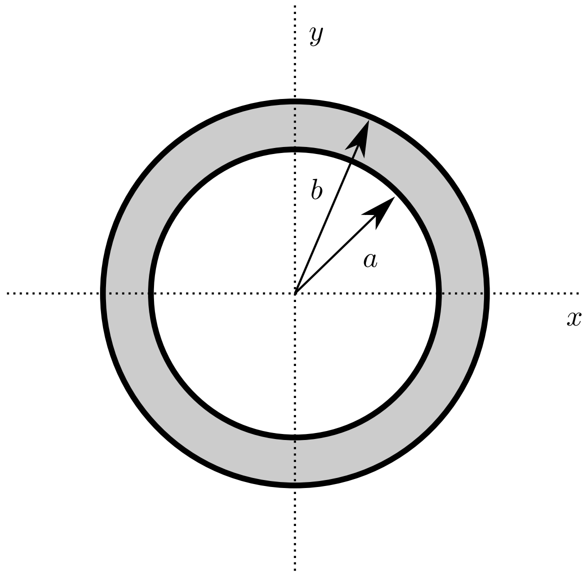
$$\frac{1}{f_2} = \frac{c/f_1}{c + \frac{gd}{c}}$$

$$f_2 = f_1\left(1 + \frac{gd}{c^2}\right)$$

While this time interval is measured in Fred's reference frame, time dilation effects are of the order $\left(\frac{gd}{c^2}\right)^2$ and can thus be ignored.

Question A4

A positive point charge q is located inside a neutral hollow spherical conducting shell. The shell has inner radius a and outer radius b; $b - a$ is not negligible. The shell is centered on the origin.



a. Assume that the point charge q is located at the origin in the very center of the shell.

i. Determine the magnitude of the electric field outside the conducting shell at $x = b$.

Solution

Conducting shell is neutral, so there is equal but opposite charge on surface $r = a$ and $r = b$. The electric field inside of a static conductor is zero, so the charge on inner surface is equal but opposite to q by Gauss's Law. Spherical symmetry requires a spherically symmetric electric field, so by Gauss's law, outside the shell, we have

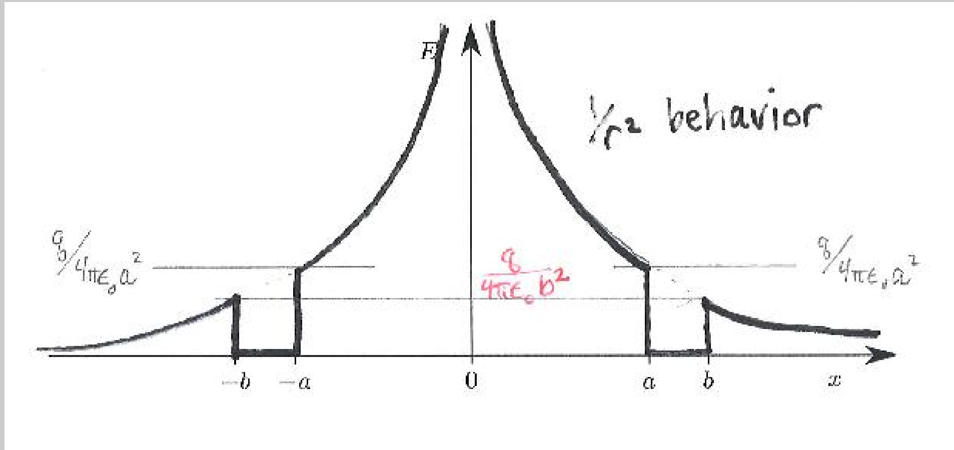
$$E(r > b) = \frac{q}{4\pi\epsilon_0 r^2}$$

and then, at $x = r = b$, we have

$$E(b) = \frac{q}{4\pi\epsilon_0 b^2}$$

ii. Sketch a graph for the magnitude of the electric field along the x axis on the answer sheet provided.

Solution



iii. Determine the electric potential at $x = a$.

Solution

The shell is a conductor, so it is an equipotential surface. This means potential at $r = a$ is same as $r = b$. For points outside the shell, spherical symmetry (and Gauss's Law) makes the problem reducible to a point charge at the origin, so

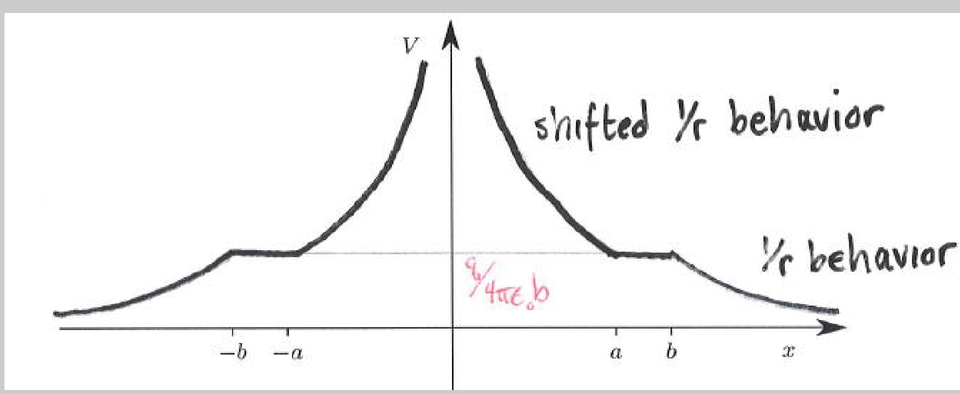
$$V(r > b) = \frac{q}{4\pi\epsilon_0 r}$$

But $V(a) = V(b)$, so $V(x = a)$ is

$$V(a) = \frac{q}{4\pi\epsilon_0 b}$$

iv. Sketch a graph for the electric potential along the x axis on the answer sheet provided.

Solution



b. Assume that the point charge q is now located on the x axis at a point $x = 2a/3$.

i. Determine the magnitude of the electric field outside the conducting shell at $x = b$.

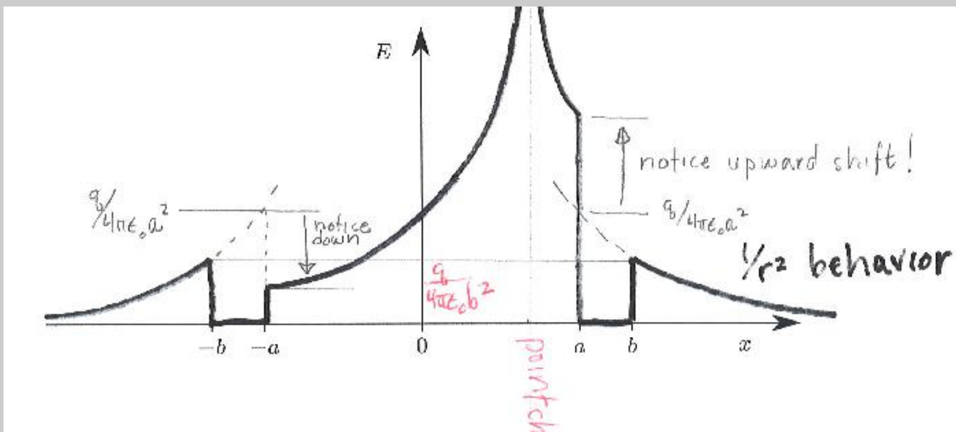
Solution

The problem (for $r > a$) maintains the spherical symmetry of above, so the answer is unchanged:

$$E(b) = \frac{q}{4\pi\epsilon_0 b^2}$$

ii. Sketch a graph for the magnitude of the electric field along the x axis on the answer sheet provided.

Solution



iii. Determine the electric potential at $x = a$.

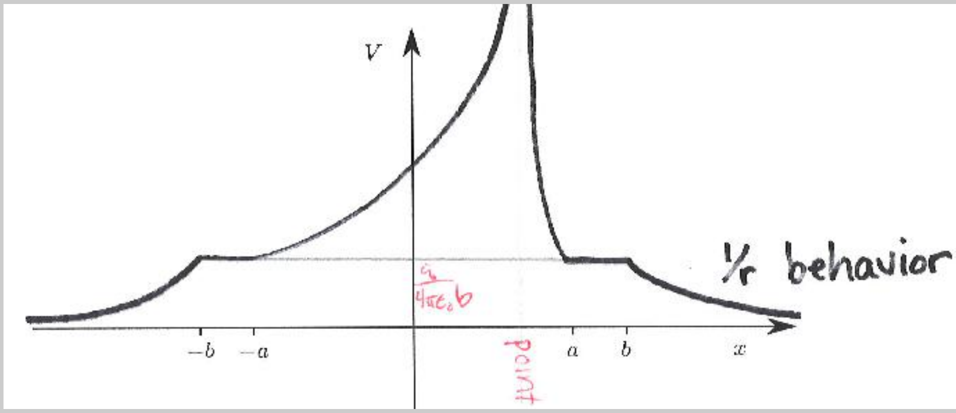
Solution

The problem (for $r > a$) maintains the spherical symmetry of above, so the answer is unchanged:

$$V(a) = \frac{q}{4\pi\epsilon_0 b}$$

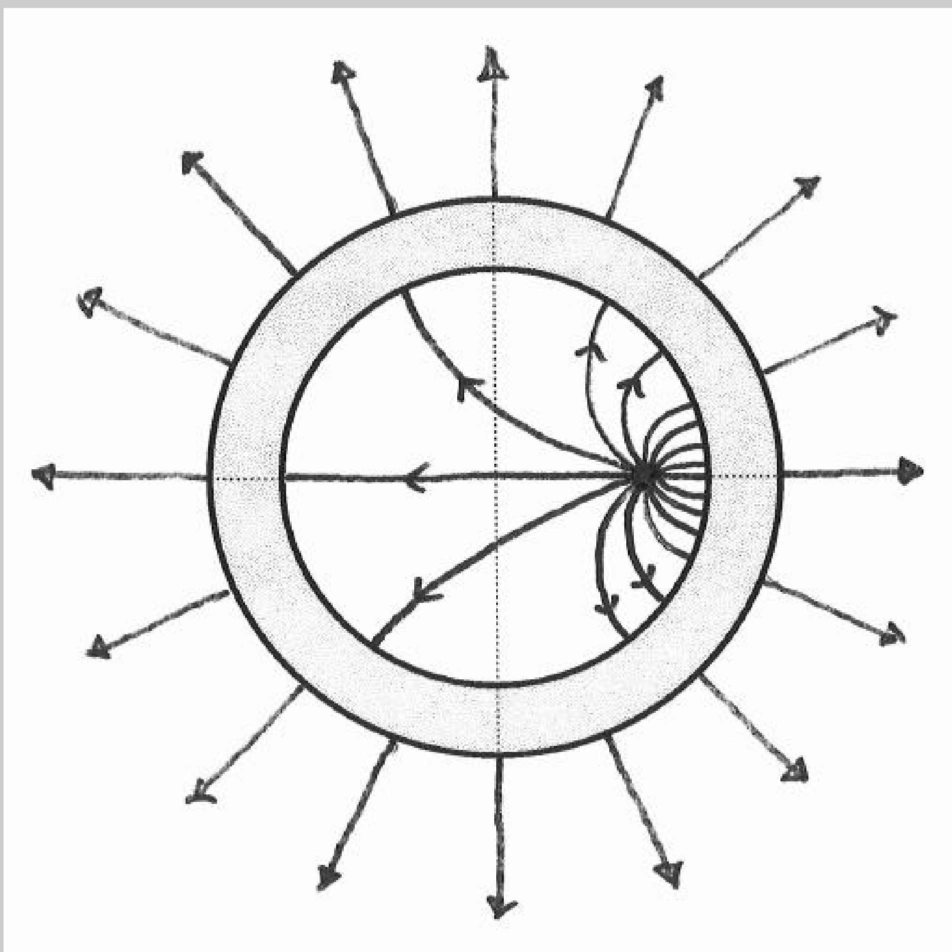
iv. Sketch a graph for the electric potential along the x axis on the answer sheet provided.

Solution



v. Sketch a figure showing the electric field lines (if any) inside, within, and outside the conducting shell on the answer sheet provided. You should show at least eight field lines in any distinct region that has a non-zero field.

Solution



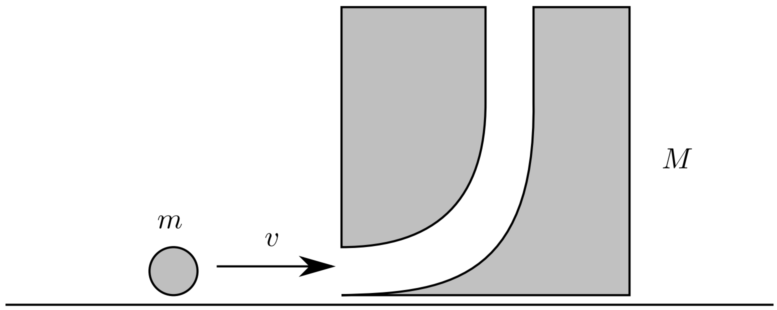
STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

A block of mass M has a hole drilled through it so that a ball of mass m can enter horizontally and then pass through the block and exit vertically upward. The ball and block are located on a frictionless surface; the block is originally at rest.



a. Consider the scenario where the ball is traveling horizontally with a speed v_0. The ball enters the block and is ejected out the top of the block. Assume there are no frictional losses as the ball passes through the block, and the ball rises to a height much higher than the dimensions of the block. The ball then returns to the level of the block, where it enters the top hole and then is ejected from the side hole. Determine the time t for the ball to return to the position where the original collision occurs in terms of the mass ratio $\beta = M/m$, speed v_0, and acceleration of free fall g.

Solution

The ball will be ejected vertically relative to the block, so the collision is effectively inelastic. This means the horizontal velocity v_1 of the block and ball after the collision will be given by

$$v_1 = \frac{m}{m+M} v_0$$

The ball will now have a vertical component to the velocity, v_2 Since there are no frictional losses, we can use conservation of energy to determine this vertical velocity. Prior to the collision, the kinetic energy is

$$E_0 = \frac{1}{2} m {v_0}^2$$

After the collision, the block carries away a kinetic energy of

$$E_1 = \frac{1}{2} M {v_1}^2$$

so the ball must have kinetic energy

$$E_2 = E_0 - E_1 = \frac{1}{2} m \left({v_1}^2 + {v_2}^2\right)$$

Equating,

$$\begin{aligned} m{v_0}^2 - M{v_1}^2 &= m\left({v_1}^2 + {v_2}^2\right) \\ m{v_0}^2 - (M+m)\left(\frac{m}{m+M}\right)^2 {v_0}^2 &= m{v_2}^2 \\ \left(1 - \frac{m}{m+M}\right){v_0}^2 &= {v_2}^2 \\ \sqrt{\frac{M}{m+M}}v_0 &= v_2 \end{aligned}$$

The time spent by the ball in the air is given by

$$t_2 = 2v_2/g$$

The distance traveled horizontally by the ball in the air is given by

$$x = v_1 t_2 = 2v_1 v_2/g$$

or

$$x = 2\frac{m}{m+M}v_0\sqrt{\frac{M}{m+M}}v_0\frac{1}{g} = \frac{2{v_0}^2}{g}\sqrt{\frac{m^2 M}{(m+M)^3}}$$

When the ball returns to the block it then is projected horizontally back toward the starting point (but it is now a distance x further away. It moves with a velocity v_3 given by the equation for an elastic collision

$$v_3 = v_0\frac{m-M}{m+M}$$

The time for the ball to return to where first collision occurs is then given by

$$t_3 = \frac{x}{|v_3|}$$

or

$$t_3 = \frac{2v_0}{g}\sqrt{\frac{m^2 M}{(m+M)^3}}\frac{m+M}{M-m}$$

or

$$t_3 = \frac{2v_0}{g}\sqrt{\frac{m^2 M}{(M+m)(M-m)^2}}$$

The total time is the sum of t_2 and t_3, or

$$\frac{2v_0}{g}\left(\sqrt{\frac{M}{m+M}} + \sqrt{\frac{m^2 M}{(M+m)(M-m)^2}}\right)$$

which can be simplified to give

$$\frac{2v_0}{g}\sqrt{\frac{M}{m+M}}\left(1 + \sqrt{\frac{m^2}{(M-m)^2}}\right)$$

or

$$\frac{2v_0}{g}\sqrt{\frac{M}{m+M}}\left(\frac{M}{M-m}\right)$$

Writing it in terms of $\beta = M/m$,

$$\frac{2v_0}{g}\sqrt{\frac{\beta}{1+\beta}}\left(\frac{\beta}{\beta-1}\right)$$

b. Now consider friction. The ball has moment of inertia $I = \frac{2}{5}mr^2$ and is originally not rotating. When it enters the hole in the block it rubs against one surface so that when it is ejected upwards the ball is rolling without slipping. To what height does the ball rise above the block?

Solution

Some of the previous work still holds true.

The ball will be ejected vertically relative to the block, so the collision is effectively inelastic. This means the horizontal velocity v_1 of the block and ball after the collision will be given by

$$v_1 = \frac{m}{m+M}v_0$$

The vertical velocity v_4 of the ball, however, is now changed from v_2. Friction slows the ball down with an impulse given by

$$\Delta p = f\Delta t$$

all the while causing the ball to rotate with a new angular momentum given by

$$L = \tau\Delta T = rf\Delta t$$

but $L = I\omega = Iv_4/r$ and $m(v_2 - v_4) = \Delta p$, so

$$Iv_4/r = mr(v_2 - v_4)$$

or, writing $I = \alpha mr^2$,

$$v_4 = \frac{v_2}{1+\alpha}$$

The vertical velocity of the ball will take it to a height

$$h = \frac{{v_4}^2}{2g}$$

so

$$h = \frac{{v_0}^2}{2g}\frac{1}{(1+\alpha)^2}\frac{M}{m+M}$$

Thankfully, the problem reduces to what we expect if $M \gg m$ and $\alpha = 0$.

Writing it in terms of $\beta = M/m$ and $\alpha = 2/5$,

$$h = \frac{{v_0}^2}{2g}\frac{25}{49}\frac{\beta}{1+\beta}$$

Question B2

In parts a and b of this problem assume that velocities v are much less than the speed of light c, and therefore ignore relativistic contraction of lengths or time dilation.

a. An infinite uniform sheet has a surface charge density σ and has an infinitesimal thickness. The sheet lies in the xy plane.

i. Assuming the sheet is at rest, determine the electric field $\vec{\mathbf{E}}$ (magnitude and direction) above and below the sheet.

Solution

From symmetry, the fields above and below the sheet are equal in magnitude and directed away from the sheet. From Gauss's Law, using a cylinder of end area A,

$$2EA = \frac{\sigma A}{\epsilon_0}$$

$$E = \sigma/2\epsilon_0$$

pointing directly away from the sheet in the z direction, or

$$\vec{\mathbf{E}} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}$$

above the sheet and

$$\vec{\mathbf{E}} = -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}$$

below the sheet

ii. Assuming the sheet is moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{x}}$ (parallel to the sheet), determine the electric field $\vec{\mathbf{E}}$ (magnitude and direction) above and below the sheet.

Solution

The motion does not affect the electric field, which is still directed away from the sheet and still has magnitude

$$E = \frac{\sigma}{2\epsilon_0}$$

iii. Assuming the sheet is moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{x}}$, determine the magnetic field $\vec{\mathbf{B}}$ (magnitude and direction) above and below the sheet.

Solution

Application of the right-hand rule indicates that (for $v > 0$) there is a magnetic field in the $-y$ direction for $z > 0$ and in the $+y$ direction for $z < 0$. From Ampere's law applied to a loop of length l normal to the x direction,

$$2Bl = \mu_0 \sigma v l$$

(In a time t, an area vtl moves through the loop, so the charge that moves is σvtl and the current is σvl.) So

$$B = \frac{\mu_0 \sigma v}{2}$$

This can also be written as

$$\vec{\mathbf{B}} = -\frac{\mu_0 \sigma v}{2}\hat{\mathbf{y}}$$

above the sheet and

$$\vec{\mathbf{B}} = \frac{\mu_0 \sigma v}{2}\hat{\mathbf{y}}$$

below the sheet

iv. Assuming the sheet is moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{z}}$ (perpendicular to the sheet), determine the electric field $\vec{\mathbf{E}}$ (magnitude and direction) above and below the sheet.

Solution

Once again the motion does not affect the electric field, which is still directed away from the sheet and still has magnitude

$$E = \frac{\sigma}{2\epsilon_0}$$

v. Assuming the sheet is moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{z}}$, determine the magnetic field $\vec{\mathbf{B}}$ (magnitude and direction) above and below the sheet.

Solution

There is no magnetic field above and below the sheet. Interestingly, there's no magnetic field at the sheet either. Consider an Amperian loop of area A in the x-y plane as the sheet passes through. The loop experiences a current of the form

$$A\sigma\delta(t).$$

But the loop also experiences an oppositely directed change in flux of the form

$$A\frac{\sigma}{\epsilon_0}\delta(t),$$

so the right-hand side of Ampere's law remains zero.

b. In a certain region there exists only an electric field $\vec{\mathbf{E}} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$ (and no magnetic field) as measured by an observer at rest. The electric and magnetic fields $\vec{\mathbf{E}}'$ and $\vec{\mathbf{B}}'$ as measured by observers in motion can be determined entirely from the local value of $\vec{\mathbf{E}}$, regardless of the charge configuration that may have produced it.

i. What would be the observed electric field $\vec{\mathbf{E}}'$ as measured by an observer moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{z}}$?

Solution

The electric field was unaffected by the motion of the sheet of charge, so the electric field in the frame of reference of the moving observer should be the same:

$$\vec{\mathbf{E}}' = \vec{\mathbf{E}}$$

ii. What would be the observed magnetic field $\vec{\mathbf{B}}'$ as measured by an observer moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{z}}$?

Solution

No magnetic field was created by the motion of the sheet of charge in the direction of the electric field, so the magnetic field in the frame of reference of the moving observer should likewise not depend on the component of the electric field in the direction of motion. When the sheet of charge was moving in the $+x$ direction, a magnetic field was created in the $-y$ direction; the observer moving in the $+x$ direction is equivalent to the sheet of charge moving in the $-x$ direction, creating a magnetic field in the $+y$ direction. That is, an electric field in the $\hat{z}$ direction causes an observer moving in the $\hat{x}$ direction to observe a magnetic field in the $\hat{y} = \hat{z} \times \hat{x}$ direction.

Previously, furthermore, the electric and magnetic fields satisfied

$$B = \frac{\mu_0 \epsilon_0}{v} E$$

Combining this with the previous equation,

$$\vec{\mathbf{B}}' = -\mu_0 \epsilon_0 \vec{\mathbf{v}} \times \vec{\mathbf{E}}$$

or, using $c^2 = 1/\mu_0 \epsilon_0$,

$$\vec{\mathbf{B}}' = -\frac{1}{c^2} \vec{\mathbf{v}} \times \vec{\mathbf{E}}$$

or in our case, where $\vec{\mathbf{v}} = v\hat{\mathbf{z}}$,

$$\vec{\mathbf{B}}' = \frac{v}{c^2} \left(E_y \hat{\mathbf{x}} - E_x \hat{\mathbf{y}} \right)$$

c. An infinitely long wire wire on the z axis is composed of positive charges with linear charge density λ which are at rest, and negative charges with linear charge density $-\lambda$ moving with speed v in the z direction.

i. Determine the electric field $\vec{\mathbf{E}}$ (magnitude and direction) at points outside the wire.

Solution

The wire as a whole is neutral, so there is no electric field outside the wire.

ii. Determine the magnetic field $\vec{\mathbf{B}}$ (magnitude and direction) at points outside the wire.

Solution

The current in the wire is λv, so Ampere's Law quickly yields

$$B = \mu_0 \frac{\lambda v}{2\pi r}$$

in the tangential direction to a circle of radius r centered on and perpendicular to the wire.

The actual direction is given by the right hand rule; since the current is in the $-z$ direction, the circular B field lines would point clockwise looking in that direction.

iii. Now consider an observer moving with speed v parallel to the z axis so that the negative charges appear to be at rest. There is a symmetry between the electric and magnetic fields such that a variation to your answer to part b can be applied to the magnetic field in this part. You will need to change the multiplicative constant to something dimensionally correct and reverse the sign. Use this fact to find and describe the electric field measured by the moving observer, and comment on your result. (Some familiarity with special relativity can help you verify the direction of your result, but is not necessary to obtain the correct answer.)

Solution

Part b indicated that a velocity boost in an electric field resulted in a magnetic field given by

$$\vec{\mathbf{B}}' = -\frac{1}{c^2}\vec{\mathbf{v}} \times \vec{\mathbf{E}}.$$

By symmetry, and insisting on a dimensionally correct answer, we assume that

$$\vec{\mathbf{E}}' = \pm\vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

is correct, give or take a sign. We were told to switch the sign, so make it positive.

Taking the cross product yields an electric field vector that points outward, with magnitude

$$E' = v\mu_0 \frac{\lambda v}{2\pi r} = \frac{\lambda}{2\pi\epsilon_0 r}\frac{v^2}{c^2}$$

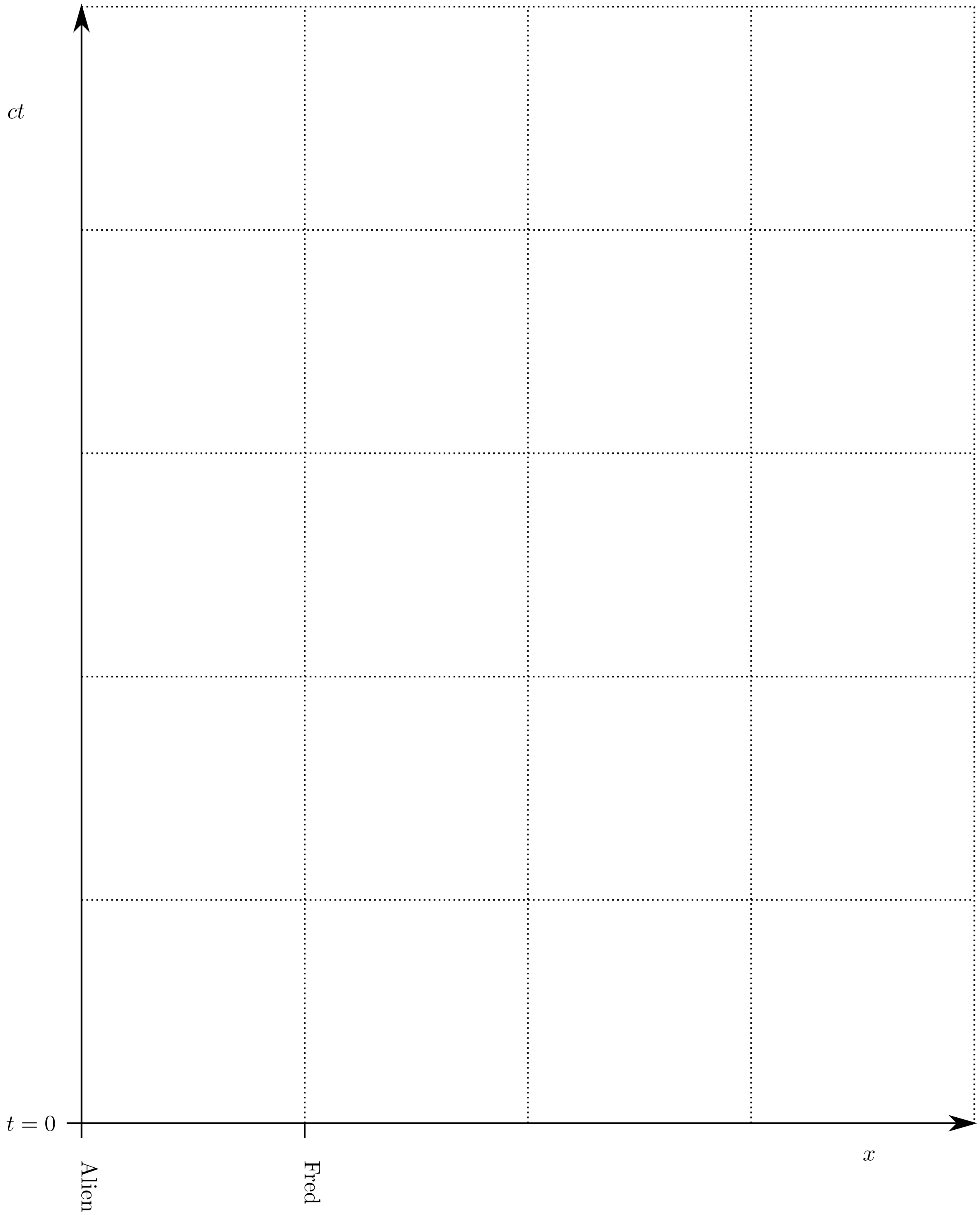
This can be attributed to length contraction of the positive charges and inverse length contraction of the (now-stationary) negative charges.

Answer Sheets

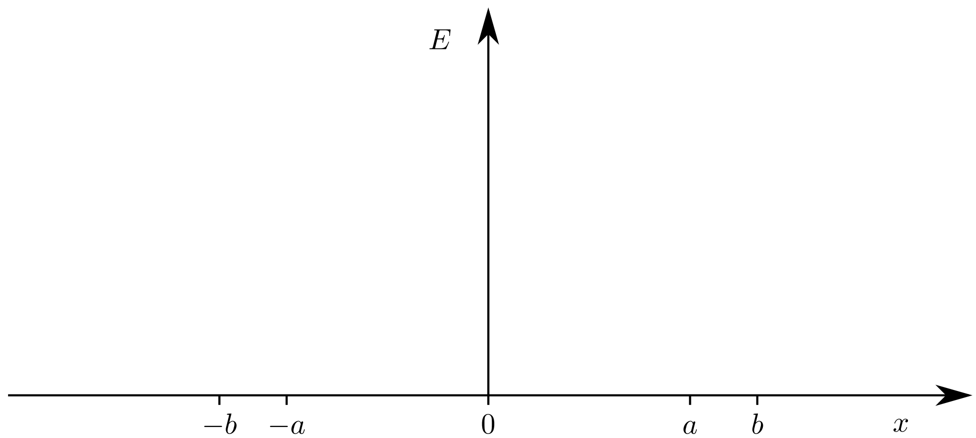
Following are answer sheets for some of the graphical portions of the test.

Answer for Part A, Question 3

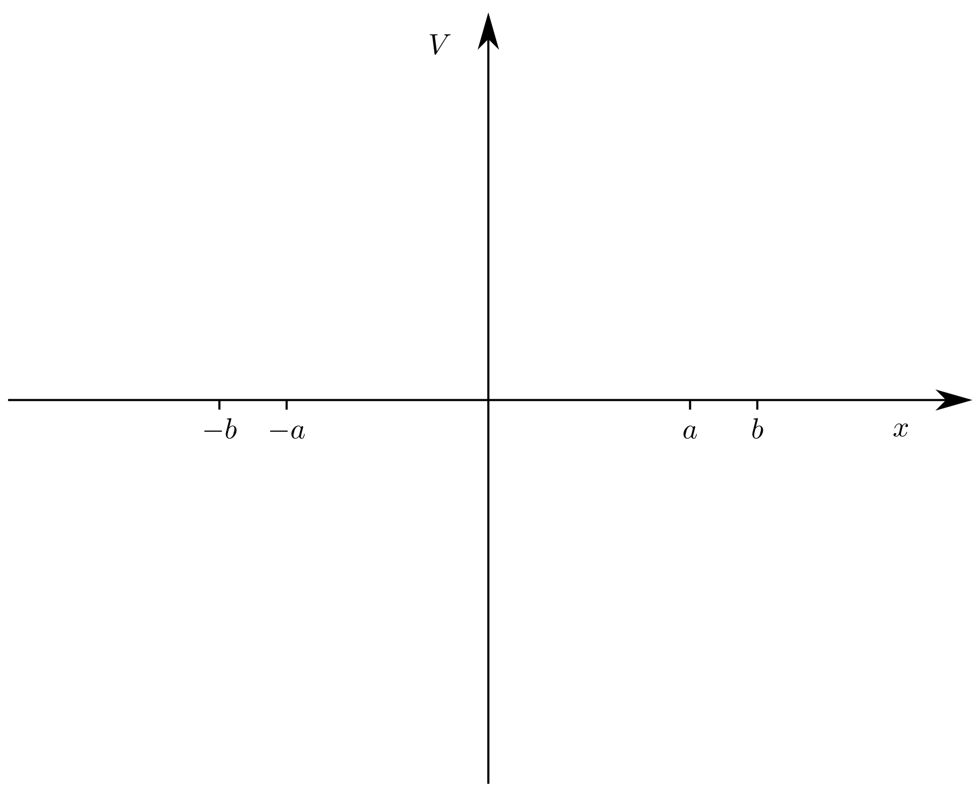
Space-time graph for accelerated rocket. The positions of Fred and the Alien at $t = 0$ are shown.



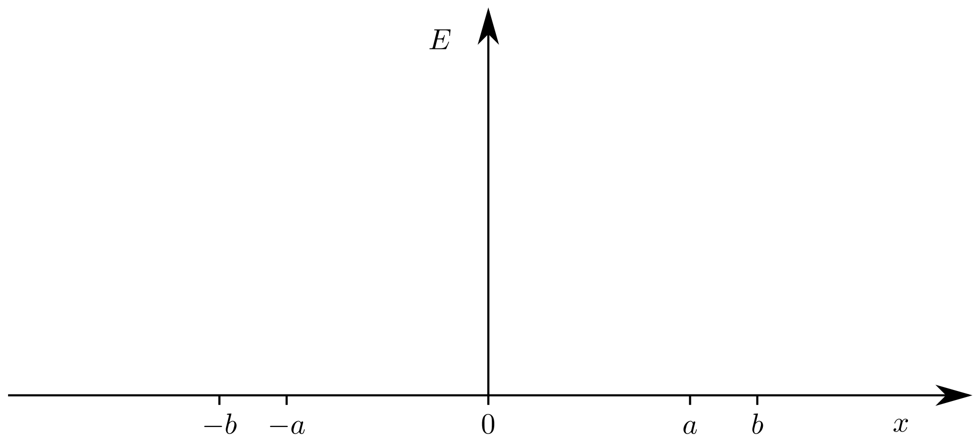
Answer for Part A, Question 4



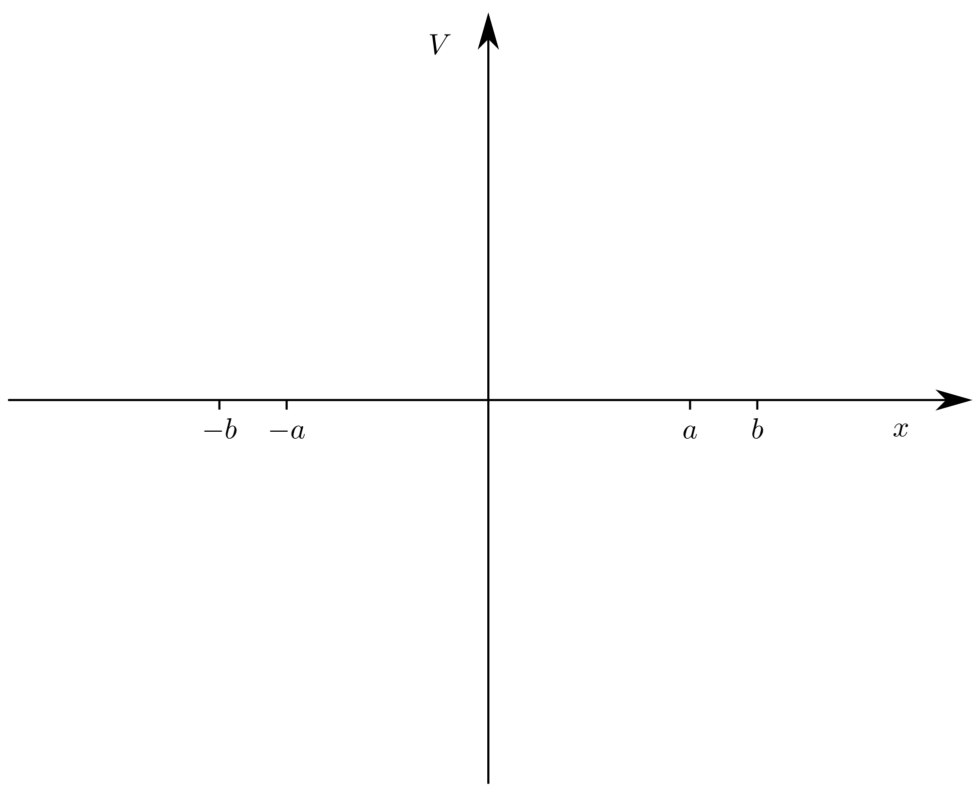
Answer for Part A, Question 4



Answer for Part A, Question 4



Answer for Part A, Question 4



Answer for Part A, Question 4