AAPT UNITED STATES PHYSICS TEAM
AIP 2008

Semifinal Exam

6 QUESTIONS - Several MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Show all your work. Partial credit will be given.
- Start each question on a new sheet of paper. Put your name in the upper right-hand corner of each page, along with the question number and the page number/total pages for this problem. For example,

  Doe, Jamie

  Prob. 1 - P. 1/3
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared.
- Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- Each of the four questions in part A are worth 25 points. Each of the two questions in part B are worth 50 points. The questions are not necessarily of the same difficulty. Good luck!
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after April 10, 2008.**

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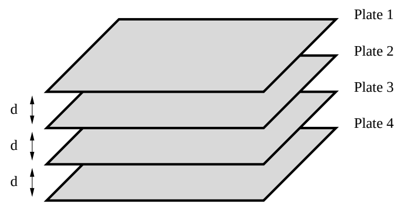
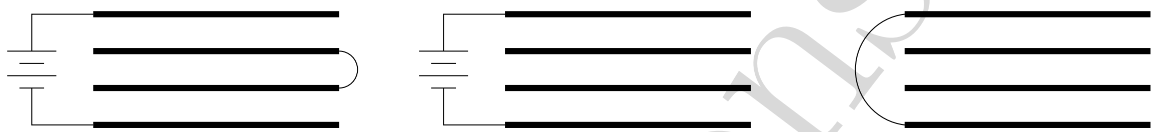
Part A

Question A1

Four square metal plates of area A are arranged at an even spacing d as shown in the diagram. (Assume that $A >> d^2$.)



Plates 1 and 4 are first connected to a voltage source of magnitude V_0, with plate 1 positive. Plates 2 and 3 are then connected together with a wire, which is subsequently removed. Finally, the voltage source attached between plates 1 and 4 is replaced with a wire. The steps are summarized in the diagram below.



(a) (b) (c)

What is the resulting potential difference between

a. Plates 1 and 2 (Call it V_1),

b. Plates 2 and 3 (Call it V_2), and

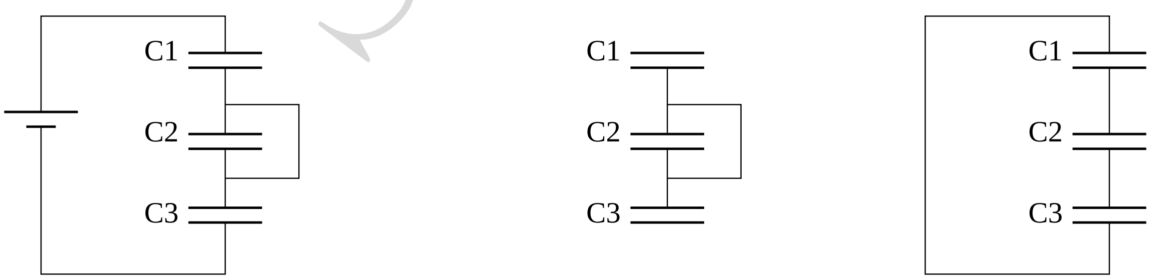
c. Plates 3 and 4 (Call it V_3).

Assume, in each case, that a positive potential difference means that the top plates is at a high potential than the bottom plate.

Solution

There are two fairly easy ways to do this problem, one rather straightforward application of capacitors, the other a more elegant, and much, much short, application of boundary conditions in electric fields.

The first method involves treating the problem as three series capacitors. Each has an identical capacitance C. The figure below then show the three steps.



Since C_2 is shorted out originally, then effectively there are only two capacitors in series, so the voltage drop across each is $V_0/2$, where the a positive potential difference means that the top plate of any given capacitor is positive. The top plate of C_1 will then have a positive charge of $q_0 = CV_0/2$. Note that this means that the bottom plate of the top capacitor will have a negative charge of $-q_0$. Removing the shorting wire across C_2 will *not* change the charges or potential drops across the other two capacitors. Removing the source V_0 will also make no difference.

Shorting the top plate of C_1 with the bottom plate of C_3 *will* make a difference. Positive charge will flow out of top plate of C_1 into the bottom plate of C_3. Also, negative charge will flow out of the bottom plate of C_1 into the top plate of C_2. The result is that C_1 will acquire a potential difference of V_1, C_2 a potential difference of V_2, and C_3 a potential difference of V_3. Let the final charge on the *top* plate of each capacitor also be labeled as q_1, q_2, and q_3.

The last figure implies that

$$V_1 + V_2 + V_3 = 0.$$

By symmetry, we have

$$V_1 = V_3.$$

so

$$2V_1 = -V_2.$$

By charge conservation between the bottom plate of C_1 and the top plate of C_2 we have

$$-q_0 = -q_1 + q_2.$$

But $q = CV$, so

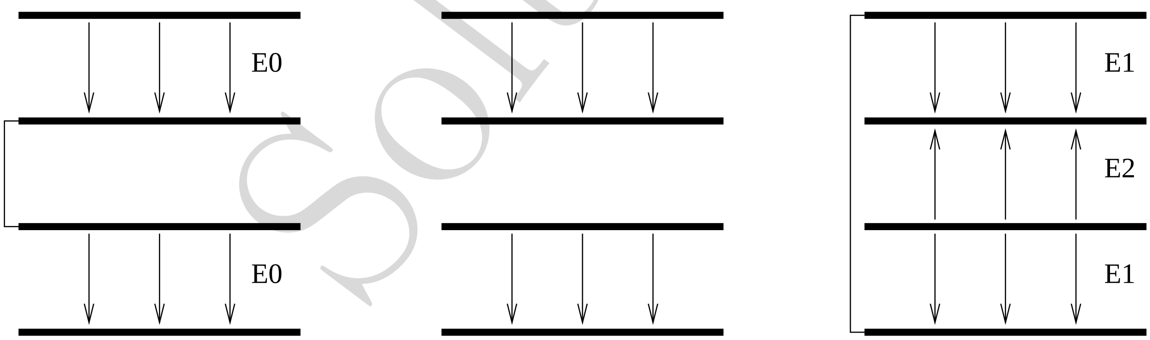
$$-\frac{1}{2}V_0 = -V_1 + V_2$$

Combining the above we get

$$\begin{aligned} -\frac{1}{2}V_0 &= \frac{1}{2}V_2 + V_2, \\ -\frac{1}{3}V_0 &= V_2. \end{aligned}$$

Finally, solving for V_1, we get $V_1 = V_0/6$.

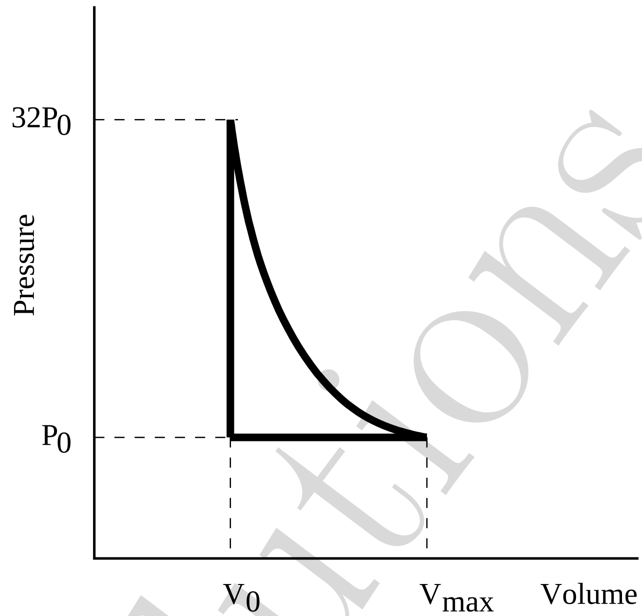
Alternatively, we could focus on the plate arrangement and the fact that across a boundary $|\Delta E_\perp| = |\sigma/\epsilon_0|$, a consequence of Gauss's Law. Also, we have, for parallel plate configurations, $|\Delta V| = |Ed|$. Since ϵ_0 and d are the same for each of the three regions, it is sufficient to simply look at the behavior of the electric fields.



In the first picture we require that $2E_0 = V_0/d$. The charge density on the second plate requires that $\Delta E = E_0$. In the last picture we have $2E_1 + E_2 = 0$, since the potential between the top plate and the bottom plate is zero. But we also have, on the second plate, $\Delta E = E_1 - E_2$. Combining, $E_0 = -\frac{1}{2}E_2 - E_2 = -\frac{3}{2}E_2$, and therefore $V_2 = -\frac{1}{3}V_0$, and $V_1 = V_0/6$.

Question A2

A simple heat engine consists of a piston in a cylinder filled with an ideal monatomic gas. Initially the gas in the cylinder is at a pressure P_0 and volume V_0. The gas is slowly heated at constant volume. Once the pressure reaches $32P_0$ the piston is released, allowing the gas to expand so that no heat either enters or escapes the gas as the piston moves. Once the pressure has returned to P_0 the the outside of the cylinder is cooled back to the original temperature, keeping the pressure constant. For the monatomic ideal gas you should assume that the specific heat capacity at constant volume is given by $C_V = \frac{3}{2}nR$, where n is the number of moles of the gas present and R is the ideal gas constant. You may express your answers in fractional form or as decimals. If you choose decimals, keep three significant figures in your calculations. The diagram below is not necessarily drawn to scale.



a. Let $V_{\max}$ be the maximum volume achieved by the gas during the cycle. What is $V_{\max}$ in terms of V_0? *If you are unable to solve this part of the problem, you may expressyour answers to the remaining parts in terms of* $V_{\max}$ *without further loss of points.*

b. In terms of P_0 and V_0 determine the heat added to the gas during a complete cycle.

c. In terms of P_0 and V_0 determine the heat removed from the gas during a complete cycle.

d. Defining efficiency e as the net work done by the gas divided by the heat added to the gas, what is the efficiency of this cycle?

e. Determine the ratio between the maximum and minimum temperatures during this cycle.

Solution

It is convenient to construct two tables and solve this problem in a manner similar to a Sudoku puzzle. Defining point 1 to be the initial point, and measuring P, V, and T in terms of P_0, V_0, and T_0, while measuring Q, W, and ΔU in terms of $nRT_0 = P_0V_0$, we have initially

Point	P	V	T
1	1	1	1
2	32	1	
3	1		

Process	Q	W	ΔU
$1 \to 2$		0	
$2 \to 3$	0		
$3 \to 1$			
net			0

The "obvious" values have been filled in: the initial conditions, and zeroes corresponding to $Q = 0$ along an adiabat, $W = 0$ along a constant volume process, and finally $\Delta U = 0$ for a net process.

The convention that will be used here is $Q + W = \Delta U$.

The ideal gas law, $PV/T = nR$, can be used to quickly determine T_2, since P/T is a constant for that process. One can then use

$$Q = C_V \Delta T$$

to find $Q_{1\to 2}$. The table values will now read

Point	P	V	T
1	1	1	1
2	32	1	32
3	1		

Process	Q	W	ΔU
$1 \to 2$	$\frac{3}{2}31$	0	
$2 \to 3$	0		
$3 \to 1$			
net			0

For the adiabatic process we have PV^γ is a constant, where $\gamma = C_P/C_V$. Students who don't know this can derive it, although it will take some time.

The derivation is straightforward enough. Along an adiabatic process, $Q = 0$, so from $Q + W = \delta U$

$$-P\,dV = \frac{3}{2}nR\,dT = \frac{3}{2}(P\,dV + V\,dP)$$

Rearranging,

$$0 = 5P\,dV + 3V\,dP$$

or

$$0 = \frac{5}{3}\frac{dV}{V} + \frac{dP}{P}$$

Integrating,

$$\text{Constant} = \frac{5}{3}\ln V + \ln P$$

which can be written in the more familiar form

$$PV^\gamma = \text{Constant}.$$

The factor of 32 was chosen so that the results are nice answers.

One can then find V_3 (and, for that matter, $V_{\max}$) by using this, and get

$$\frac{V_3}{V_2} = \left(\frac{P_2}{P_3}\right)^{1/\gamma} = (32)^{3/5} = 8$$

Putting this in the table, and then quickly applying the ideal gas law to find T_3 then enables the finding of $Q_{3\to 1}$, since along this process

$$Q = C_P \Delta T = \frac{5}{2}nR\Delta T.$$

The tables now look like

Point	P	V	T
1	1	1	1
2	32	1	32
3	1	8	8

Process	Q	W	ΔU
$1 \to 2$	$\frac{3}{2}31$	0	
$2 \to 3$	0		
$3 \to 1$	$-\frac{5}{2}7$		
net			0

It is now possible to determine Q_{net} and, from $Q + W = \Delta U$, W_{net}.

So the tables now look like

Point	P	V	T
1	1	1	1
2	32	1	32
3	1	8	8

Process	Q	W	ΔU
$1 \to 2$	$\frac{3}{2}31$	0	
$2 \to 3$	0		
$3 \to 1$	$-\frac{5}{2}7$		
net	29	-29	0

The negative net work reflects that the gas does work on the outside world. The efficiency of the process is then

$$e = \frac{29}{93/2} = \frac{58}{93}$$

It isn't much more work to fill in all of the values for both tables, a task that each student ought be able to do. One key point will be process $3 \to 1$, where $W = -P\Delta V$. the the rest are filled in by applications of $Q + W = \Delta U$.

Point	P	V	T
1	1	1	1
2	32	1	32
3	1	8	8

Process	Q	W	ΔU
$1 \to 2$	$\frac{3}{2}31$	0	$\frac{3}{2}31$
$2 \to 3$	0	-36	-36
$3 \to 1$	$-\frac{5}{2}7$	7	$-\frac{3}{2}7$
net	29	-29	0

Question A3

A certain planet of radius R is composed of a uniform material that, through radioactive decay, generates a net power P. This results in a temperature differential between the inside and outside of the planet as heat is transfered from the interior to the surface.

The rate of heat transfer is governed by the thermal conductivity. The thermal conductivity of a material is a measure of how quickly heat flows through that material in response to a temperature gradient. Specifically, consider a thin slab of material of area A and thickness Δx where one surface is hotter than the other by an amount ΔT. Suppose that an amount of heat ΔQ flows through the slab in a time Δt. The thermal conductivity k of the material is then

$$k = \frac{\Delta Q}{\Delta t} \frac{1}{A} \frac{\Delta x}{\Delta T}.$$

It is found that k is approximately constant for many materials; assume that it is constant for the planet.

For the following assume that the planet is in a steady state; temperature might depend on position, but does not depend on time.

a. Find an expression for the temperature of the surface of the planet assuming blackbody radiation, an emissivity of 1, and *no* radiation incident on the planet surface. You may express your answer in terms of any of the above variables and the Stephan-Boltzmann constant σ.

b. Find an expression for the temperature difference between the surface of the planet and the center of the planet. You may express your answer in terms of any of the above variables; you do *not* need to answer part (a) to be able to answer this part.

Solution

For the first question, apply the Boltzmann equation, and

$$P = \sigma A T_s^4$$

where A is the surface area of the planet, and T_s the temperature at the center. Then

$$T_s = \left(\frac{P}{4\pi\sigma R^2} \right)^{1/4}$$

For the second question, it is reasonable to assume that the temperature depends on the distance form the center only. Then the definition of k gives for a spherical shell of thickness dr

$$k = \frac{\Delta Q}{\Delta t} \frac{1}{4\pi r^2} \frac{dr}{dT}.$$

The heat through the shell depends on the power radiated from within the shell. Since the planet is uniform, this depends on the volume according to

$$\frac{\Delta Q}{\Delta t} = P \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} = P \frac{r^3}{R^3}$$

so that rearrangement yields

$$dT = \frac{P}{4\pi k R^3} r \, dr$$

Integrating between the center and the surface,

$$\Delta T = \frac{P}{8\pi k R},$$

which could be used to find the temperature of the interior.

Question A4

A tape recorder playing a single tone of frequency f_0 is dropped from rest at a height h. You stand directly underneath the tape recorder and measure the frequency observed as a function of time. Here $t = 0$s is the time at which the tape recorder was dropped.

t (s)	f (Hz)
2.0	581
4.0	619
6.0	665
8.0	723
10.0	801

The acceleration due to gravity is $g = 9.80$ m/s^2 and the speed of sound in air is $v_a = 340$ m/s. Ignore air resistance. You might need to use the Doppler shift formula for co-linear motion of sources and observers in still air,

$$f = f_0 \frac{v_a \pm v_o}{v_a \pm v_s}$$

where f_0 is the emitted frequency as determined by the source, f is the frequency as detected by the observer, and v_a, v_s, and v_o are the speed of sounds in air, the speed of the source, and the speed of the observer. The positive and negative signs are dependent upon the relative directions of the source and the observer.

a. Determine the frequency measured on the ground at time t, in terms of f_0, g, h, and v_a.

b. Verify graphically that your result is consistent with the provided data.

c. What (numerically) is the frequency played by the tape recorder?

d. From what height h was the tape recorder dropped?

Solution

The position of the tape recorder above the ground at a time t is given by

$$y = h - \frac{1}{2}gt^2$$

and the speed of the tape recorder is given by

$$v_s = -gt$$

The observer "hears" the sound emitted from the tape recorder a time δt earlier, since it takes time for the sound to travel to the listener. In this case,

$$y = v_a \delta t$$

So at time t the listener is hearing the tape recorder when it had emitted at time $t' = t - \delta t$, or

$$t' = t - \frac{h}{v_a} + \frac{g}{2v_a}(t')^2$$

Solve this for t', first by rearranging,

$$\frac{g}{2}(t')^2 - v_a t' + (v_a t - h) = 0$$

and the by applying the quadratic formula

$$t' = \frac{v_a \pm \sqrt{{v_a}^2 + 2gh - 2gv_a t}}{g}.$$

This might not look right, but in the limit of small h and large v_a, it does reduce to the expected $t' = t$ if one keeps the negative result.

Consequently,

$$v_s = \sqrt{{v_a}^2 + 2gh - 2gv_at} - v_a$$

gives the velocity of that source had when it emitted the sound heard at time t. This result is negative, indicating motion down, and toward the observer, so one must use the positive sign in the denominator of the Doppler shift formula.

Applying the Doppler shift formula,

$$f = f_0 \frac{v_a}{\sqrt{{v_a}^2 + 2gh - 2gv_at}}$$

which, in the limit of large v_a and small h, reduces to

$$f = f_0 \left(1 + \frac{g}{v_a}t\right)$$

Keeping to the correct expression, we can rearrange it as

$$\frac{1}{f^2} = \frac{1}{{f_0}^2}\left(1 + \frac{2gh}{{v_a}^2} - \frac{2g}{v_a}t\right)$$

which would graph as a straight line by plotting t horizontally and $1/f^2$ vertically. The slope of the line would yield

$$-\frac{2g}{v_a{f_0}^2}$$

while the vertical intercept would yield

$$\frac{1}{{f_0}^2}\left(1 + \frac{2gh}{{v_a}^2}\right)$$

Using this hint, a quick table of data to consider graphing would be

t (s)	f (Hz)	$1/f^2$ ($\times 10^{-6}$ s^2)
2.0	581	2.96
4.0	619	2.61
6.0	665	2.26
8.0	723	1.91
10.0	801	1.56

The slope is -1.75×10^{-7} s.

Then

$$f_0 = \sqrt{\frac{2(9.8)}{(1.75 \times 10^{-7})(340)}}\text{Hz} = 574\text{Hz}$$

The intercept is 3.31×10^{-6} s^2. This yields a height in meters given by

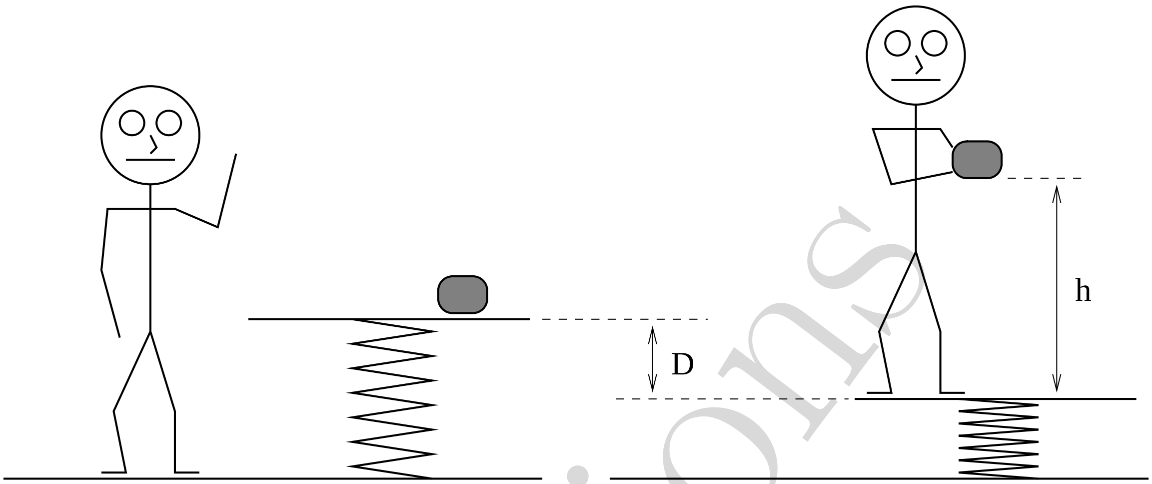
$$h = (340)\frac{(3.31)}{(0.175)} - \frac{(340)^2}{2(9.8)} = 533$$

Clearly, an impressive building; and a more impressive tape player, that it could be heard from such a distance!

Part B

Question B1

A platform is attached to the ground by an ideal spring of constant k; both the spring and the platform have negligible mass. Sitting on the platform is a rather large lump of clay of mass m_c. You then gently step onto the platform, and the platform settles down to a new equilibrium position, a vertical distance D below the original position. Assume that your mass is m_p.



a. You then pick up the lump of clay and hold it a height h above the platform. Upon releasing the clay the you and the platform will oscillate up and down; you notice that the clay strikes the platform after the platform has completed exactly one oscillation. Determine h in terms of any or all of k, D, the masses m_p and m_c, the acceleration of free fall g, and any necessary numerical constants. You must express your answer in the simplest possible form.

b. Assume the resulting collision between the clay and the platform is completely inelastic. Find the ratio of the amplitude of the oscillation of the platform before the collision (A_i) and the amplitude of the oscillations of the platform after the collision (A_f). Determine A_f/A_i in terms of any or all of k, D, the masses m_p and m_c, the acceleration of free fall g, and any necessary numerical constants.

c. Sketch a graph of the position of the platform as a function of time, with $t = 0$ corresponding to the moment when the clay is dropped. Show one complete oscillation *after* the clay has collided with the platform.

d. The above experiment is only possible if the ratio m_c/m_p is smaller than some critical value r_c, otherwise the clay will hit the platform *before* one complete oscillation. An *estimate* for the value of the critical ratio r_c can be obtained by assuming the clay hits the platform after exactly one-half of an oscillation. Assume that h is the same as is determined by part (a), and use this technique to determine r_c in terms of any or all of k, D, the mass m_p, the acceleration of free fall g, and any necessary numerical constants.

e. Is this estimate for r_c too large or too small? You *must* defend your answer with an appropriate diagram.

Solution

Stepping on the platform will lower it a distance D. This means that the spring constant of the platform spring is given by

$$kD = m_p g.$$

If the lump of clay is removed, then the equilibrium position of the platform would rise a distance A given by

$$kA = m_c g.$$

This would also be the amplitude of the oscillations after the clay is released, so

$$A_i = \frac{m_c g}{k}$$

The frequency of oscillation of the plate without the clay is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m_p}}$$

The time for a complete oscillation is

$$T = 2\pi\sqrt{\frac{m_p}{k}}$$

If the clay falls a distance h, then

$$h = \frac{1}{2}gT^2 = 2\pi^2 g\frac{m_p}{k} = 2\pi^2 H.$$

When the plate is at the stating point it is at rest. The clay will hit it with a speed given by

$$v_0 = gT.$$

Conservation of momentum in an inelastic collision will then result in a final speed of the clay + platform system of

$$v_f = v_0\frac{m_c}{m_c + m_p}.$$

The kinetic energy just after collision will be

$$K = \frac{1}{2}(m_c + m_p){v_f}^2.$$

So the amplitude of the resulting oscillations will be given by

$$\frac{1}{2}k{A_f}^2 = \frac{1}{2}(m_c + m_p){v_f}^2$$

or

$$A_f = v_f\sqrt{\frac{m_c + m_p}{k}}$$

Gluing stuff together

$$\begin{aligned}
\frac{A_f}{A_i} &= \frac{v_f}{A_i}\sqrt{\frac{m_c + m_p}{k}}, \\
&= \frac{v_0}{A_i}\frac{m_c}{m_c + m_p}\cdot\sqrt{\frac{m_c + m_p}{k}}, \\
&= \frac{gTk}{m_c g}\frac{m_c}{m_c + m_p}\cdot\sqrt{\frac{m_c + m_p}{k}}, \\
&= T\sqrt{\frac{k}{m_p + m_c}}
\end{aligned}$$

And then, combining with our previous expression for T,

$$\begin{aligned}
\frac{A_f}{A_i} &= 2\pi\sqrt{\frac{m_p}{k}}\sqrt{\frac{k}{m_p + m_c}}, \\
&= 2\pi\sqrt{\frac{m_p}{m_p + m_c}}
\end{aligned}$$

If, instead, the clay manages to hit the platform at the top of an oscillation, then the distance the clay would fall would only be

$$h - 2A$$

and the time required would be

$$\frac{T}{2} = \pi\sqrt{\frac{m_p}{k}}.$$

Then

$$h - 2A = \frac{1}{2}g\left(\frac{T}{2}\right)^2,$$

where that complicated looking thing is actually

$$\frac{1}{4}h.$$

So

$$\begin{aligned} h &= \frac{1}{4}h + 2A, \\ \frac{3}{8}h &= A, \\ \frac{3}{4}\pi^2 D &= A. \end{aligned}$$

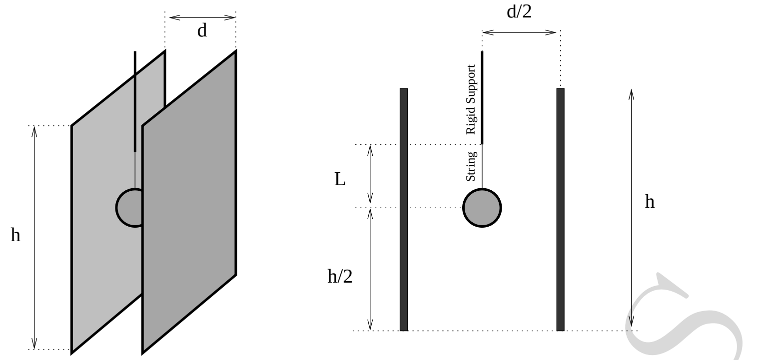
But from the very first two equations,

$$\frac{m_c}{m_p} = \frac{A}{D} = \frac{3}{4}\pi^2.$$

In this scenario the clay passes the platform three times: once at the highest point, once some distance further on, and once when the at the lowest point. That means that if A were smaller, it might be possible to find a value of A such that the clay just barely touches the platform once before hitting the platform at the bottom. Consequently, this is an *over* estimate for A, and an overestimate for m_c, and therefore, and overestimate for the ratio. A larger ratio would guarantee collision, the actual critical ratio would be smaller than this.

Question B2

Consider a parallel plate capacitor with the plates vertical. The plates of the capacitor are rigidly supported in place. The distance between the plates is d. The plates have height h and area $A >> d^2$. Assume throughout this problem that air resistance may be neglected; however, the force of gravity cannot be neglected.



a. A metal ball with a mass M and a charge q is suspended from a string that is tied to a rigid support. When the capacitor is not charged, the metal ball is located at the center of the capacitor (at a distance $d/2$ from both plates and at a height $h/2$ above the bottom edge of the plates. If instead a constant potential difference V_0 is applied across the plates, the string will an angle θ_0 to the vertical when the metal ball is in equilibrium.

   i. Determine θ_0 in terms of the given quantities and fundamental constants.

   ii. The metal ball is then lifted until it makes an angle θ to the vertical where θ is only slightly greater than θ_0. The metal ball is released from rest. Show that the resulting motion is simple harmonic motion and find the period of the oscillations in terms of the given quantities and fundamental constants.

   iii. When the ball is at rest in its equilibrium position, the string is cut. What is the maximum value for V_0 so that the ball will *not* hit one of the plates before exiting? Express your answer in terms of the given quantities and fundamental constants.

b. Suppose instead that the ball of mass M and charge q is released from rest at a point halfway between the plates at a time $t = 0$. Now, an AC potential difference $V(t) = V_0 \sin \omega t$ is also placed across the capacitor. For what range of angular frequencies will the ball not hit either plate before it falls, under the influence of gravity, out of the region between the plates? Consider only two scenarios: either $g >> h\omega^2$ or $g << h\omega^2$. Express your answer in terms of the given quantities and fundamental constants.

Solution

The electric field between the plates is given by

$$E = V_0/d$$

There are then three forces on the hanging ball: electrostatic, gravitational, and the tension in the string. They are

$$F_E = qE = qV_0/d$$

and

$$F_g = Mg$$

and then tension F_T.

In equilibrium, the string must make an angle such that the horizontal and vertical components of the tension balance the electrostatic and gravitational forces. Consequently,

$$\tan\theta_0 = \frac{F_E}{F_g} = \frac{qV_0}{Mgd}$$

The tension in the string is given by

$$F_T{}^2 = F_g{}^2 + F_E{}^2.$$

A simple pendulum oscillates with a period according to

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where g is the acceleration of free fall. In this case, if the string were cut, the *instantaneous* acceleration of the ball would be F_T/M, so the period of small oscillations would be

$$T = 2\pi\sqrt{\frac{ML}{F_T}}$$

This is not a particularly pretty thing to substitute for, but yields

$$T = 2\pi\sqrt{\frac{ML}{\sqrt{M^2g^2 + V_0{}^2q^2/d^2}}}$$

which can be simplified, somewhat, to give

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \left(\frac{V_0 q}{Mgd}\right)^2\right)^{-\frac{1}{4}}$$

An astute reader would recognize a few trigonometric identities and write

$$T = 2\pi\sqrt{\frac{L\cos\theta_0}{g}}$$

which is certainly much more compact.

Upon cutting the string the ball will move in a straight line, tangent to the angle the string originally made. It won't move with constant speed. So it will leave the region between the plates a distance

$$x_1 = (L + h/2)\tan\theta_0$$

away from the center line. Set this equal to $d/2$, the condition that it hits one of the plates, and solve for V_0:

$$\begin{aligned} \frac{d}{2} &= (L + h/2)\frac{qV_0}{Mgd} \\ V_0 &= \frac{Mgd^2}{(2L + h)q} \end{aligned}$$

If the ball is instead released from rest in the oscillating field then it will experience an oscillating force in the x direction given by

$$F_E = \frac{V_0}{d}\sin\omega t.$$

Hence, the x component of the acceleration will be given by

$$a_x = \frac{qV_0}{Md}\sin\omega t.$$

The ball is released from rest, so this can be directly integrated to give the x component of the velocity

$$v_x = \frac{qV_0}{Md\omega}\left(1 - \cos\omega t\right).$$

This can be directly integrated to find the position relative to the center, which we will define as $x = 0$:

$$x = \frac{qV_0}{Md\omega}\left(t - \frac{1}{\omega}\sin\omega t\right)$$

The ball hits one of the plates if this value exceeds $d/2$ while it is still in the region between the plates. Solving for equality,

$$\frac{Md^2\omega}{2qV_0} = t - \frac{\sin\omega t}{\omega}$$

is the concern. t is given by the equation of a falling object, or

$$\frac{h}{2} = \frac{1}{2}gt^2,$$

so $t = \sqrt{h/g}$.

Unfortunately, this can't be solved for ω.

Consider first the condition that $g << h\omega^2$. Then it is reasonable to neglect $1/\omega$ compared to $\sqrt{g/h}$, and the equation simplifies to

$$\frac{Md^2\omega}{2qV_0} = \sqrt{g/h}$$

with solution

$$\omega_1 = \frac{2qV_0}{Md^2}\sqrt{g/h}$$

At higher frequencies the ball will miss, falling out of the plates before it hits either side.

Consider instead the condition that $g >> h\omega^2$. This is a slowly oscillating field, and one can then approximate the sine function as

$$\sin\omega t \approx \omega t - \frac{1}{6}\omega^3 t^3.$$

yielding

$$\frac{Md^2\omega}{2qV_0} = \frac{1}{6}\omega^2 t^3$$

or

$$\omega_2 = \frac{3Md^2}{qV_0}\sqrt{h^3/g^3}$$

as the second critical frequency. At lower frequencies the ball will not hit either side before it falls out of the region between the plates.

Note that the product of these two angular frequencies is

$$\theta_1\theta_2 = 6\frac{h}{g}$$

Optical Society of America Bonus Question

Researchers have a developed a lens made of liquid. The spherical lens consists of a droplet of transparent liquid resting on a electrically controllable surface. When the voltage of the surface is changed, the droplet changes itself shape; it either tries to "ball-up" more strongly or it becomes flatter. Figure 1 is a sketch of the liquid lens and several parameters that describe it, including the thickness of the lens (t), the radius of curvature of the top surface (R) and the contact angle (θ), which represents the angle between the flat surface beneath the droplet and the tangent to the curved surface at the point of contact.

a. When a certain voltage is applied, both the contact angle and lens thickness increase (and the lens becomes more curved). In this case, is the liquid attracted or repelled by the surface?

b. Express the contact angle as a function of R and t.

c. The total volume of the liquid lens is an important parameter because as the liquid lens changes shape, its volume is conserved. Calculate the volume of the lens as a function of R and t.

d. Use your result to part (b) to eliminate the variable t from your expression for the volume and find $V(R, \theta)$.

e. By changing the voltage on the control surface, the contact angle, θ, can be changed, which in turn changes the focal length of the lens, f. The lensmaker's formula can be used to calculate the focal length and is given by

$$\frac{1}{f} = (n_{\text{liquid}} - n_{\text{air}}) \left(\frac{1}{R_1} - \frac{1}{R_2} \right),$$

where n_{liquid} and n_{air} are the refractive indices of the liquid in the lens and air around it, and R_1 and R_2 are the radii of curvature of the two surfaces of the lens. In figure 1, R_1 is the curved face and R_2 is the flat face. Use the lensmaker's formula to calculate the focal length of the lens in terms of the total volume of the liquid, the contact angle, and the relevant refractive indices.

Sidenote: liquid lenses are interesting because they are electrically controllable, variablefocus lenses that can be very compact. People are working on putting them into cell phone cameras for ultra-compact zoom lenses. For more information on this type of liquid lens, see T. Krupenkin, S. Yang, and P. Mach, "Tunable liquid microlens," Appl. Phys. Lett. 82, 316-318 (2003).