Damped Oscillations of a Free Piston in a Gas-Filled Cylinder

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ABSTRACT

If a cylinder is capped off by a sliding piston, we have a situation analogous to a mass on a spring. With suitable idealizations\(^1\) the mass on the spring is undamped and it will oscillate forever if initially displaced from equilibrium. With other suitable idealizations\(^2\) will the piston similarly oscillate forever if initially displaced? No! Unlike the solid bonds inside a spring, the gas molecules are mobile and so the analog is not exact. In fact, the motion of a piston in a gas-filled cylinder is always damped. However, the damping is weak and so the frequency of oscillation in a Rüchardt experiment closely approximates the undamped frequency.

\(^1\)The mass hangs vertically in vacuum from a Hookean spring attached to a rigid support.

\(^2\)The piston has no friction with the cylinder; the gas is ideal with no viscosity or turbulence; there is vacuum on the other side of the piston; and the piston and cylinder have zero thermal conductivity and heat capacity.
The bulk of the gas has time-varying pressure $P$, absolute temperature $T$, and volume $V = Ax$. However, the gas atoms next to the piston (occupying the hatched slice of negligible volume compared to $V$) exert dynamic pressure $\tilde{P}$ on the piston.
A gas atom of mass $\mu$ and upward velocity $u$ makes an elastic collision with the piston of mass $m$ and upward velocity $v$, where $m \gg \mu$ and $u \gg v$.

kinetic theory \( \Rightarrow \tilde{P} = P \left( 1 - v \sqrt{\frac{8M}{\pi RT}} \right) \)

COUPLED EQUATIONS OF MOTION

Newton’s second law (N2L) for the piston

$$\tilde{P}A - mg = ma \implies \frac{d^2 x}{dt^2} = \frac{nRT}{mx} \left(1 - \frac{dx}{dt} \sqrt{\frac{8M}{\pi RT}}\right) - g$$

first law of thermodynamics (T1L) for the gas

$$-\tilde{P}Adx = \frac{3}{2}nRdT \implies \frac{dT}{dt} = \frac{2T}{3x} \left(\frac{dx}{dt} \sqrt{\frac{8M}{\pi RT}} - 1\right) \frac{dx}{dt}$$

Solve them simultaneously for $x(t)$ and $T(t)$.
NUMERICAL SOLUTION

**given**

- $T_i = 300$ K
- $A = 10$ cm$^2$
- $m = 10$ kg
- $M = 83.8$ g/mol
- $x_i = 1$ m
- $v_i = -2$ m/s

**derived**

- $P_i = mg / A = 9.8$ kPa
- $n = mgx_i / RT_i = 0.0393$ mol
- $x_f = 1.08$ m
- $T_f = 324$ K
- $S_f = 64.1$ mJ/K
The increase in $S$ confirms that the damping is irreversible.

$$TdS = dU + PdV \quad \Rightarrow \quad S = \frac{3}{2} nR \ln \frac{T}{T_i} + nR \ln \frac{x}{x_i}$$
UNDERDAMPED OSCILLATOR MODEL

\[ x(t) = x_f - X e^{-bt/2m} \sin(\omega t + \phi) \]

Rüchardt prediction \( \omega = \sqrt{\frac{\gamma P_f A^2}{m V_f}} = \sqrt{\frac{5nRT_f}{3mx_f^2}} \approx 3.89 \text{ rad/s} \)

damping \(-bv = (\tilde{P} - P)A \Rightarrow b = P_f A \sqrt{\frac{8M}{\pi RT_f}} = \frac{n}{x_f} \sqrt{\frac{8MRT_f}{\pi}} \approx 0.872 \text{ kg/s} \)

get \( X = 0.522 \text{ m} \) and \( \phi = 0.157 \text{ rad} \) from fitting \( x \) and \( dx/dt \) to \( x_i \) and \( v_i \)
In excellent agreement with numerical solution of the coupled equations.
FINAL COMPRESSION RATIO

What is $h_f$ after the oscillations have died away when $m_{\text{weight}}$ is suddenly placed on the piston?
substitute N2L: \( P_iA = m_{\text{piston}}g \) and \( P_fA = (m_{\text{piston}} + m_{\text{weight}})g \)

into T1L: \( \frac{3}{2} P_i Ah_i + (m_{\text{piston}} + m_{\text{weight}})gh_i = \frac{3}{2} P_f Ah_f + (m_{\text{piston}} + m_{\text{weight}})gh_f \)

to get

\[
\frac{h_f}{h_i} = 1 - \frac{0.6}{1 + \frac{m_{\text{piston}}}{m_{\text{weight}}}}
\]

so that one cannot compress the gas to less than 40% of its initial volume even if the added weight is infinite!

Contrast that with a reversible adiabatic compression described by \( P_i h_i^{5/3} = P_f h_f^{5/3} \) so that \( h_f \to 0 \) when the added weight (and hence the final pressure) is infinite.
