# Spinning Charged Bodies and the Linearized Kerr Metric

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## Abstract

The physics of the Kerr metric of general relativity (GR) can be understood qualitatively by analogy with the potentials of spinning charged spheres in electrodynamics (E&M). We make this correspondence explicit by comparing the Lagrangian for test particle motion in E&M with a spinning spherical source to the Lagrangian for a test particle in GR under the influence of a linearized limit of the Kerr metric. The interpretation of Kerr as the metric appropriate to spinning massive bodies then emerges as a simple replacement of mass for charge in the E&M case.

## I. INTRODUCTION

In E&M, as with all field theories, we have two sets of physical laws: 1. The field equations (Maxwell) that tell us how to generate $\vec{E}$ and $\vec{B}$ (or equivalently $V$ and $\vec{A}$) given a source, and 2. An equation of motion (Lorentz) that tells us how test particles respond to fields. GR is no different – Einstein's equations gives us the connection between sources and fields, and the equation of motion for test particles is just the geodesic equation describing the "straightest possible lines" in a curved space-time.

$$
\begin{array}{lll}
 & \text{E\&M} & \text{GR} \\
\text{Fields} & \begin{array}{l} \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0} \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \end{array} & G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \\
\text{Particles} & L = \frac{1}{2} m v^2 - q(V - \vec{v}\cdot\vec{A}) & L = -mc\sqrt{\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu}.
\end{array}
\tag{1}
$$

The "field" in GR is the metric $g_{\mu\nu}$ and the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is a nonlinear combination of $g_{\mu\nu}$, and its first and second derivatives (more complicated than the left-hand side of Maxwell's equations, but in much the same spirit). While E&M has $\rho$ and $\vec{J}$ as sources for $V$ and $\vec{A}$, the metric in GR is generated by the full stress tensor of matter, just an expanded notion of what can generate fields.

For the equations of motion, we have the Lagrangian associated with the Lorentz force law for E&M parametrized by time $t$ (so that $\vec{v} = \frac{d\vec{x}}{dt}$), and the geodesic Lagrangian for GR where $x^\mu$ is the coordinate four-vector with $x^0 = ct$ and the dots refer to derivatives with respect to the proper time $\tau$. Indeed, if we take $g_{\mu\nu} = \eta_{\mu\nu}$, the Minkowski metric, we reproduce the (special) relativistic Lagrangian for a free particle.

Our goal is to take a specific distribution of charge in E&M, find the Lagrangian governing test particle motion there, and compare it to the Lagrangian for the Kerr metric of GR. In order to make the comparison, we will express the geodesic equation in terms of the coordinate time (as the parameter of the motion) and take a slow-motion limit, then input Kerr in linearized form as the metric.

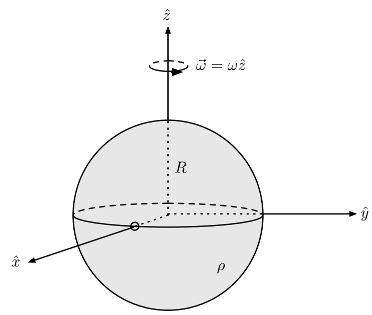


FIG. 1: A uniformly charged spinning sphere with radius $R$.

## II. A CHARGED SPINNING SPHERE

A sphere of radius $R$ with constant charge density $\rho$ spinning about the $\hat{z}$ axis with uniform angular velocity $\omega$ has well-known potentials[1]. For $r > R$,

$$V = \frac{\rho R^3}{3\epsilon_0 r} \tag{2}$$

$$\vec{A} = \frac{\mu_0 \omega \rho R^5}{15} \frac{\sin\theta}{r^2} \hat{\phi}, \tag{3}$$

in SI units. With this set of potentials, in spherical coordinates, the Lagrangian reads

$$L = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2 \right) - \frac{q\rho R^3}{3\epsilon_0 r} + \frac{\mu_0 \omega \rho R^5}{15} \frac{\sin^2\theta}{r} \dot{\phi}, \tag{4}$$

and variation with respect to $(r, \theta, \phi)$ will reproduce the Lorentz force law for this distribution.

The configuration is static (we are not considering the spin-up of the charged body – for us, it has been spinning forever with constant $\omega$), and the potential $V$ is just that of a uniformly charged sphere. We know, from our early physics education, that the Newtonian potential for a sphere of uniform density would take the form (although with different constants) of $V$ with charge density $\rho$ interpreted as mass density. But what about the other term in the Lagrangian? The magnetic vector potential has no analogue in Newtonian gravity which is a scalar theory, there is no notion that "mass current" changes the motion of test bodies.

Suppose that we take the electrostatic-Newtonian correspondence:

$$\frac{q\rho R^3}{3\epsilon_0 r} = \frac{qQ}{4\pi\epsilon_0 r} \longrightarrow \frac{GmM}{r} \tag{5}$$

at face value and apply it to the magnetic term in (4), this would give us the Lagrangian for a theory of gravity that included the "spin" of the central body as a source. Again, this is absent in Newtonian theory, so for now, the procedure is speculative. The above suggests that the correct replacement is $\frac{qQ}{4\pi\epsilon_0} \longrightarrow GmM$, so for the magnetic portion, we have

$$\frac{q\mu_0\omega\rho R^5}{15}\frac{\sin^2\theta}{r}\dot{\phi} = \frac{q\omega R^2 Q}{5r(4\pi\epsilon_0 c^2)}\sin^2\theta\dot{\phi} \longrightarrow \frac{m\omega R^2 MG}{5rc^2}\sin^2\theta\dot{\phi}, \tag{6}$$

representing the "predicted" gravitational coupling of a spinning massive central body to a test body.

## III. LINEARIZED KERR

The Einstein equations in vacuum (away from sources) reduce to $R_{\mu\nu} = 0$ for the Ricci tensor. There are only a few known solutions to this equation, but in terms of (astro)physical relevance, the Kerr solution[2] is the most useful. The Kerr metric in Boyer-Lindquist[3,4] coordinates reads:

$$\begin{aligned} g_{\mu\nu} &\doteq \begin{pmatrix} -\left(1-\frac{2MGr}{c^2\rho^2}\right) & 0 & 0 & -\frac{2aMGr\sin^2\theta}{c^3\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{2aMGr\sin^2\theta}{c^3\rho^2} & 0 & 0 & \left(\left(r^2+\left(\frac{a}{c}\right)^2\right)+\frac{2a^2MGr\sin^2\theta}{c^4\rho^2}\right)\sin^2\theta \end{pmatrix} \\ \rho^2 &\equiv r^2 + \left(\frac{a}{c}\right)^2\cos^2\theta \\ \Delta &\equiv \left(\frac{a}{c}\right)^2 - 2\frac{MGr}{c^2} + r^2. \end{aligned} \tag{7}$$

for $x^\mu \doteq (ct, r, \theta, \phi)$. This metric describes a space-time which is not flat, and therefore, the coordinates, while we have written them with familiar names, do not represent the usual spherical coordinates. There are two parameters in the metric: $a$ and $M$, whose physical significance will become clear as we go. For now, it is interesting to note that $a = M = 0$ reproduces the spherical Minkowksi metric of special relativity, and $a = 0$ gives Schwarzschild (what about $M = 0$?). It is also relatively clear that for large $r$, the above reduces to flat space. This metric does indeed have vanishing Ricci tensor, and therefore is an exact solution of Einstein's equations in vacuum.

Because we want to compare with our electrostatic Lagrangian, written in terms of a Euclidean three-dimensional space in spherical coordinates with time as a parameter, we expand the above metric in powers of the fundamental lengths: $\frac{a}{c}$ and $\frac{MG}{c^2}$ – that is, we are taking $\frac{a}{c} \ll r$ and $\frac{MG}{c^2} \ll r$. Under this assumption:

$$g_{\mu\nu} \approx \underbrace{\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}}_{\equiv \eta_{\mu\nu}} + \begin{pmatrix} -\frac{2MG}{c^2 r} & 0 & 0 & -\frac{2aMG\sin^2\theta}{c^3 r} \\ 0 & \frac{2MG}{c^2 r} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2aMG\sin^2\theta}{c^3 r} & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

This metric is only an approximate solution to the vacuum Einstein equations, there are corrections of order $\left(\frac{a}{c}\right)^2$ and $\left(\frac{MG}{c^2}\right)^2$. We have given up exactness in favor of interpretability – the form of (8) allows us to view the coordinates $(r, \theta, \phi)$ as the usual flat spherical ones, that's the role of the Minkowski metric as the dominant term.

Now we want to use the linearized metric (8) in the geodesic Lagrangian to understand how test bodies move. Our approach will be to keep terms appropriate to the linearization procedure, assume that the test body motion is slow (compared to $c$) and parametrize the motion by the coordinate time rather than proper time. The end result will be a Lagrangian in which the second term in (8) (representing deviation from flat space) appears as an effective potential which we can compare with (4).

The action for geodesic deviation reads:

$$S = -mc \int \sqrt{\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu} d\tau = -mc \int \sqrt{\frac{dx^\mu}{dt} g_{\mu\nu} \frac{dx^\nu}{dt} \left(\frac{dt}{d\tau}\right)^2} \frac{d\tau}{dt} dt = -mc \int \sqrt{\frac{dx^\mu}{dt} g_{\mu\nu} \frac{dx^\nu}{dt}} dt \tag{9}$$

which gives us the geodesic Lagrangian using $t$ as the parameter for the test body motion. In terms of general metric components, we have

$$L = -mc \sqrt{-g_{00} c^2 - 2 g_{0j} c \frac{dx^j}{dt} - \frac{dx^j}{dt} g_{jk} \frac{dx^k}{dt}} \tag{10}$$

where we let the $j$ and $k$ indices run from $1 \rightarrow 3$ (the spatial components), and note that $\frac{dx^0}{dt} = c$.

Using (8) as the metric, the Lagrangian reads

$$L = -mc \sqrt{\left(1 - \frac{2MG}{c^2 r}\right) c^2 + 4 \frac{aMG \sin^2\theta}{r^2 c} \frac{r\dot{\phi}}{c} - c^2 \left(1 + \frac{2MG}{c^2 r}\right) \left(\frac{\dot{r}}{c}\right)^2 - \left(\frac{r\dot{\theta}}{c}\right)^2 c^2 - c^2 \sin^2\theta \left(\frac{r\dot{\phi}}{c}\right)^2}. \tag{11}$$

To expand this, we use our linearization assumption and keep only terms of order $\frac{a}{rc}$, $\frac{MG}{c^2 r}$ and $\frac{a}{rc}\frac{MG}{c^2 r}$, where the metric is still a valid vacuum solution. In addition, we keep only linear and quadratic velocities, this limits us to test bodies that move much less than the speed of light. Under these assumptions, the Lagrangian is approximately:

$$L \approx \underbrace{-mc^2}_{ignorable} + \underbrace{\frac{mMG}{r} - 2\frac{amMG\sin^2\theta}{rc^2}\dot{\phi}}_{potential} + \underbrace{\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right)}_{kinetic}, \tag{12}$$

where the first term is a constant that will not show up in the equations of motion, the second term is an effective potential and the third term is the usual kinetic energy.

Compare this Lagrangian with (4) – the shared kinetic term tells us we are talking about motion in a three-dimensional flat space. The potential above has a Newtonian component that we expect, and a term proportional to $\dot{\phi}$ which is new. From the Newtonian term, we learn that $M$ appearing in (7) is the central body's mass. The new term is precisely of the form predicted by our electrostatic example. Referring to (6), if we associate (modulo some purely numerical factors) $a \sim \omega R^2$, the linearized Kerr metric includes a contribution that we can interpret as the spin of a massive body. The $a$ parameter has units of angular momentum per unit mass, and its presence in the Lagrangian tells us that GR predicts, at this linearized level (and in the slow-motion approximation) the existence of a "gravito-magnetic" potential that couples to test particles in the same way as the magnetic contribution from spinning charged spheres.

## IV. SUMMARY

In introductory physics, Newtonian gravity is usually a student's first exposure to $\frac{1}{r^2}$ forces. In addition, it is a natural place to discuss the spherical symmetry of sources and the connection between fields and forces. When the Coulomb field is introduced later on, during E&M, it is often compared to the Newtonian spherically symmetric force law. During a more advanced course in E&M, students learn about the magnetic vector potential, and so it is natural to ask if the simplest stationary configuration in E&M that supports both electric and magnetic potentials, namely the charged spinning sphere, has any natural cousin on the gravitational side.

The prime motivation for general relativity is the idea that Newtonian gravity requires modification, and looking for corrections from E&M provides a nice symmetry to the original, freshman physics notion that gravity and electrostatics have something in common. That they do at the linearized GR level is well known[5], of course, but the very explicit parallel treatment provided here is meant to be accessible to students of E&M, before the details of the linearized Einstein equations have been presented. In addition, the parameters of the Kerr metric can be given real physical meaning by this direct analogy, again, before the metric has been discussed explicitly as a vacuum solution.

## V. CLASS ROOM DISCUSSION

Following are some questions that can be posed to the students, and some ideas for additional discussion that can be used to introduce related topics in GR that follow naturally from the analogy presented here:

- We assumed, based on the form of (8), that we could interpret the coordinates as flat. That approach will not reproduce the correct numerical expression for perihelion precession in a linearized Schwarzschild limit, one must keep the non-flat radial coordinate. What are the deficiencies in the linearized Kerr setting?

- How can one visualize a coordinate like the Schwarzschild $r$? Why are diagrams of orbital motion relevant? These lead naturally to a discussion of asymptotic flatness and observers.

- For E&M, we know that vacuum solutions to the field equations are constrained by boundaries at infinite and some region enclosing the source. Vacuum Einstein equations have the same feature, and the E&M approach here makes this point.

- A general treatment of linearization and coordinate choices at the level of Einstein's equation can stem from this example.

- Given the similarities of E&M and GR, why is a vector theory of gravity insufficient?

- The geodesics of Kerr can also be treated from here, particularly the equatorial case[6,7].

- Spinning test particles are often encountered in E&M, and with the static example given here, it is relatively simple to couple test-body spin, which suggests the coupling in GR (GPB experiment and Lense-Thirring precession follow).

- The linearized limit here lacks the additional Carter's constant of full Kerr, where did it go?

We have gone through the parallels "forward" here, predicting a corrective term and then verifying it. But one could also start with the spinning charged sphere, and demand that there be a gravitational analogue to the magnetic term. Then the Kerr metric, once derived, appears more familiar, and $a$ and $M$ which show up as constants of integration in the derivation can be given immediate interpretation.

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