Relativity in the Global Positioning System

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Fundamental Principles

- Principle of Inertia
  - The laws of physics are the same in all inertial frames of reference.
- Constancy of the speed of light
  - The speed of light, *c*, is a constant independent of the motion of the source (or of the observer); (it is *c* in all inertial frames).
- Principle of Equivalence ("weak form")
  - Over a small region of space and time, the fictitious gravitational field induced by acceleration cannot be distinguished from a real gravitational field due to mass.

Changes in point of view (reference frames)

In studying relativity, one must be willing to adopt different points of view--that is, different reference frames. The physical phenomena don't change, but our description of them does change.

Example 1: the gulf stream curves toward the east as it flows north. For an observer fixed on the rotating earth, this is due to the "Coriolis force."

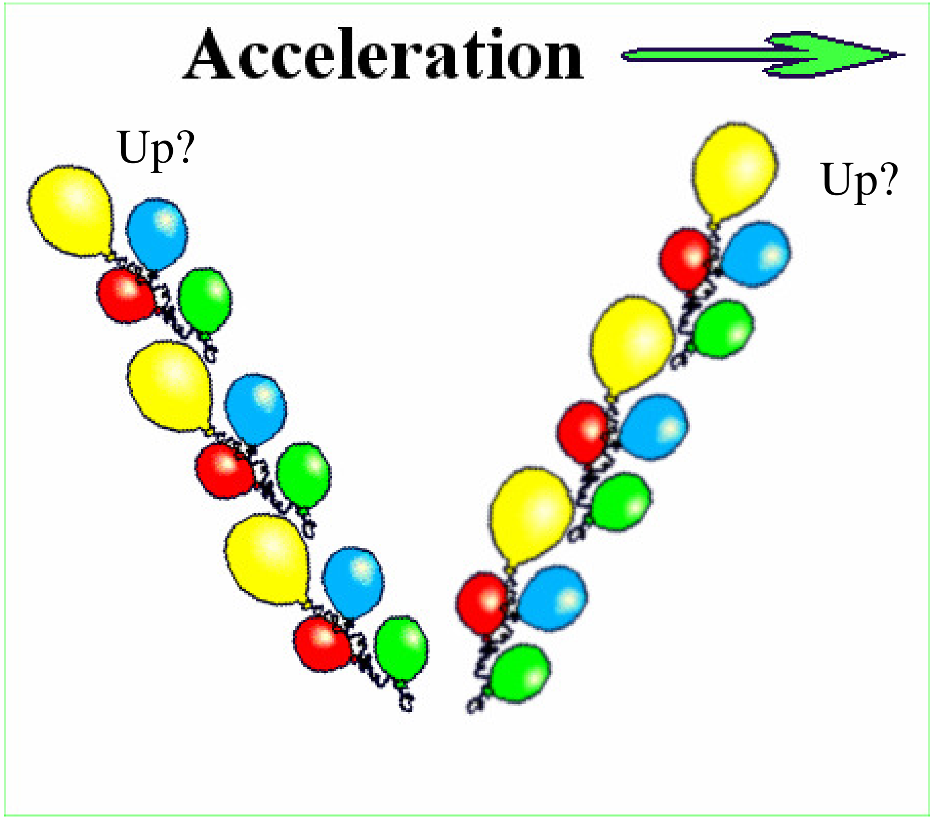
To an observer in a local, freely falling non-rotating frame attached to earth's center, this is due to conservation of angular momentum.

Example 2: A pendulum in an accelerating car points backwards. From the point of view of someone in the car, the force of gravity points slightly downwards and slightly backwards.

From the point of view of someone on the ground, the pendulum bob is accelerated forwards by a component of tension in the string that points slightly forwards and upwards.

Question.

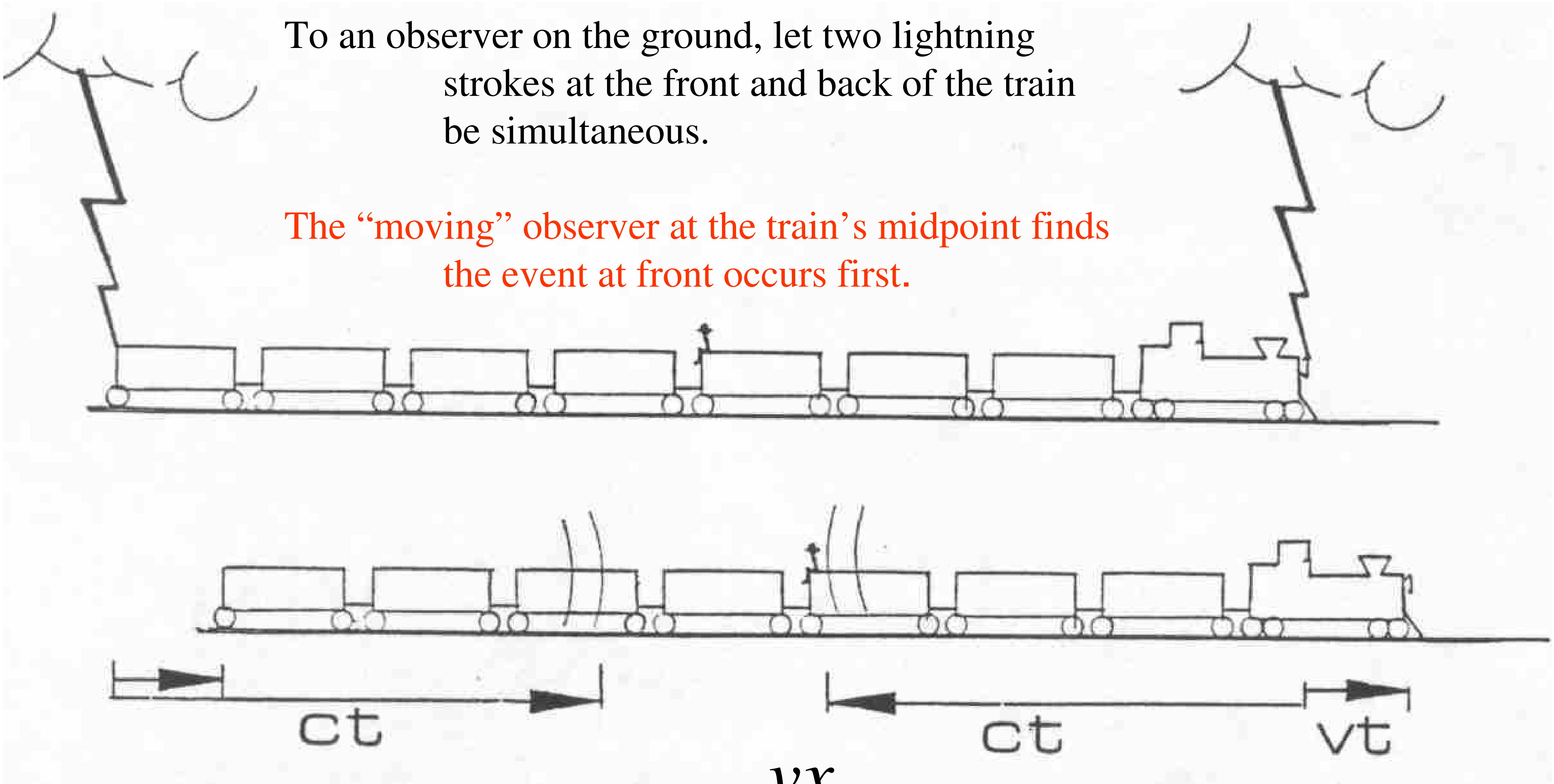
In an accelerating vehicle, if up is indicated by the direction of the string that holds down a helium balloon, what direction is "up"?



Relativity of Simultaneity

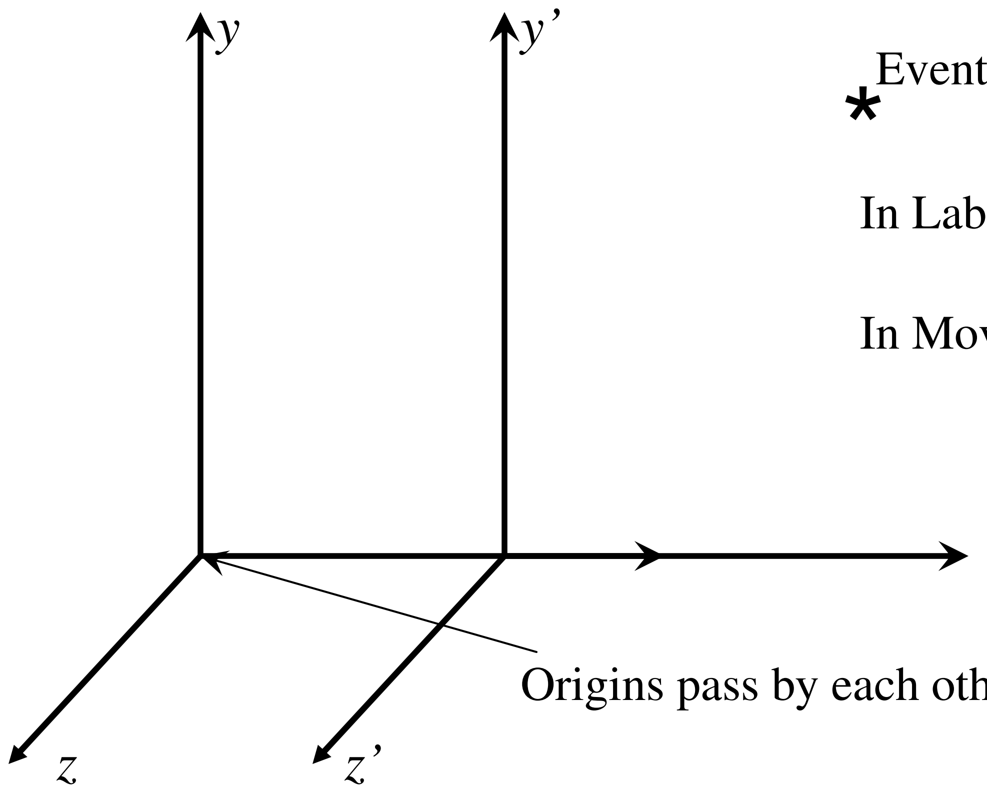
To an observer on the ground, let two lightning strokes at the front and back of the train be simultaneous.

The "moving" observer at the train's midpoint finds the event at front occurs first.



$$t' \approx t - \frac{vx}{c^2}$$

Notation--”Lab” and “Moving” Frames



In Lab Frame: $\{\ t,\ x\ ,y,\ z\ \}$

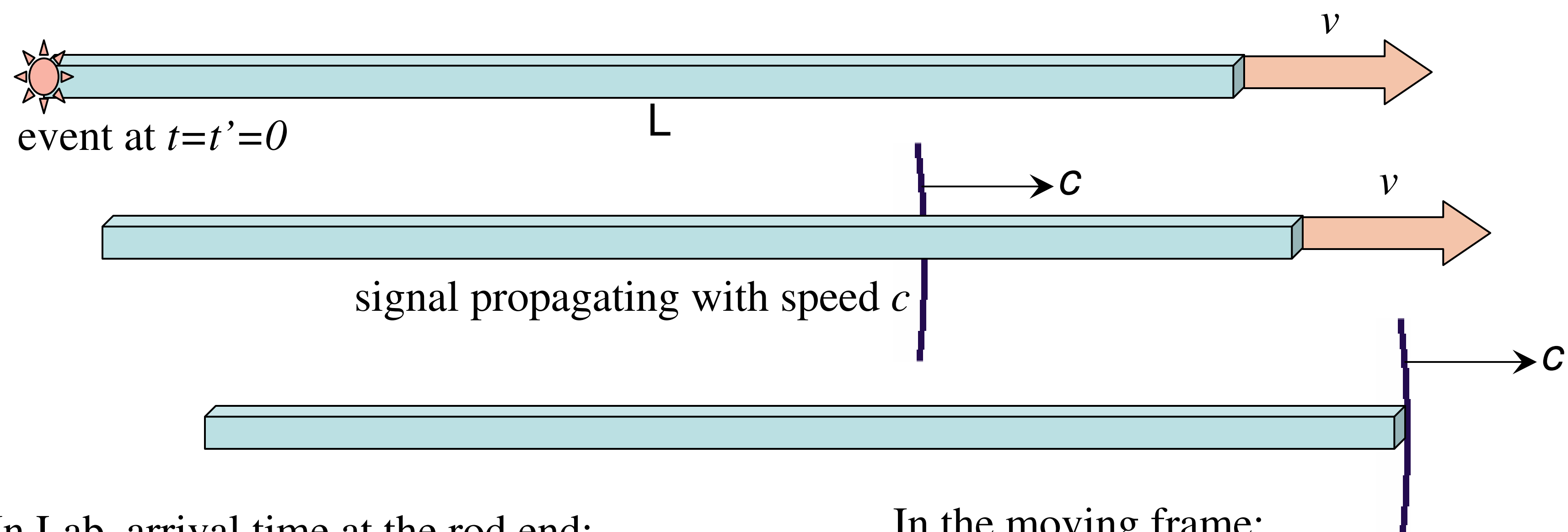
In Moving Frame: $\{\ t',\ x',\ y',\ z'\ \}$

Origins pass by each other at $t = t' = 0$.

Breakdown of simultaneity

In a given inertial frame, it's OK to take differences of velocities and obtain a velocity *difference* greater than c.

Example: Let a rod of length $L=x$ move in the positive x-direction with speed v. Light emitted at $t=0$ from the left end of the rod travels to the right end.



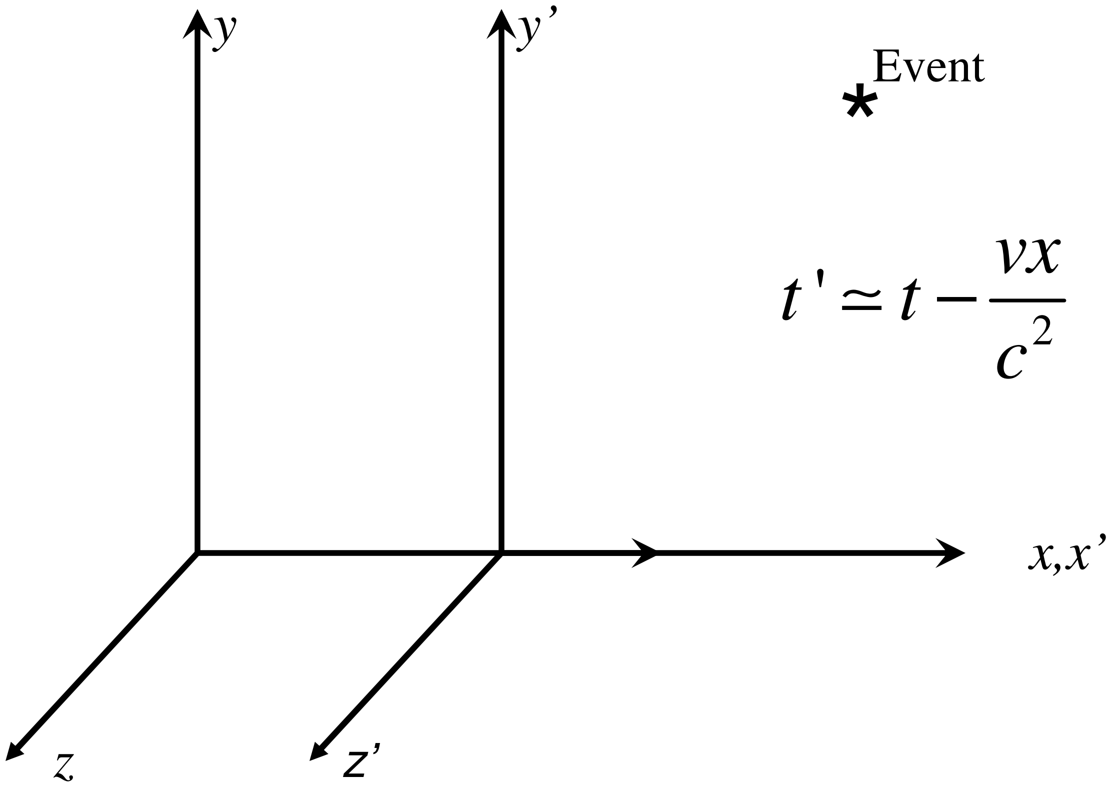
In Lab, arrival time at the rod end:

$$t = \frac{L}{(c-v)} = \frac{L}{c(1-v/c)} \simeq \frac{L}{c} + \frac{vL}{c^2};$$

In the moving frame:

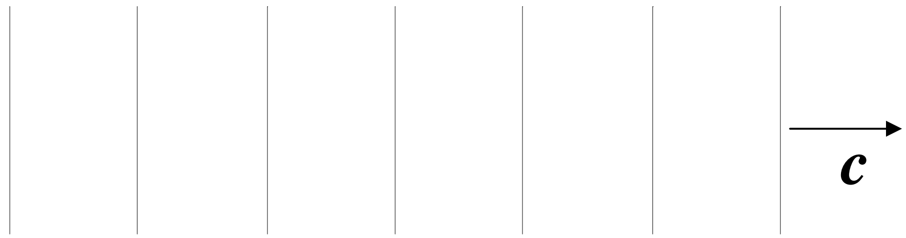
$$t' = \frac{L}{c}; \quad t' = t - \frac{vL}{c^2} = t - \frac{vx}{c^2}$$

Breakdown of simultaneity



$$t' \simeq t - \frac{vx}{c^2}$$

Relation Between Doppler Effect and Relativity of Simultaneity



Wavefronts are marked simultaneously by the "non-moving" observer:

$$t = 0$$

$$x = 0 \quad \lambda \quad 2\lambda \quad 3\lambda \ldots\ldots n\lambda \ldots$$

Moving observer says the wavefronts are marked differently:

$$t' = -\frac{vn\lambda}{c^2}$$

So the wavefront at $x = \lambda \ (n = 1)$ needs to move an additional distance

$$c\frac{v\lambda}{c^2} = \frac{v}{c}\lambda$$

before it gets into the right position to be marked at t' = 0. To the moving observer, the wavelength is:

$$\lambda' = \lambda + \frac{v}{c}\lambda = (1 + \frac{v}{c})\lambda.$$

Doppler frequency shift

$$c = f\lambda, \quad \ln c = \ln f + \ln \lambda.$$

So when lambda increases, the frequency decreases. Taking the differential of the logarithm function gives

$$\frac{\Delta f}{f} = -\frac{\Delta \lambda}{\lambda} = -\frac{v}{c}.$$

Equivalence Principle and Gravitational Frequency Shifts

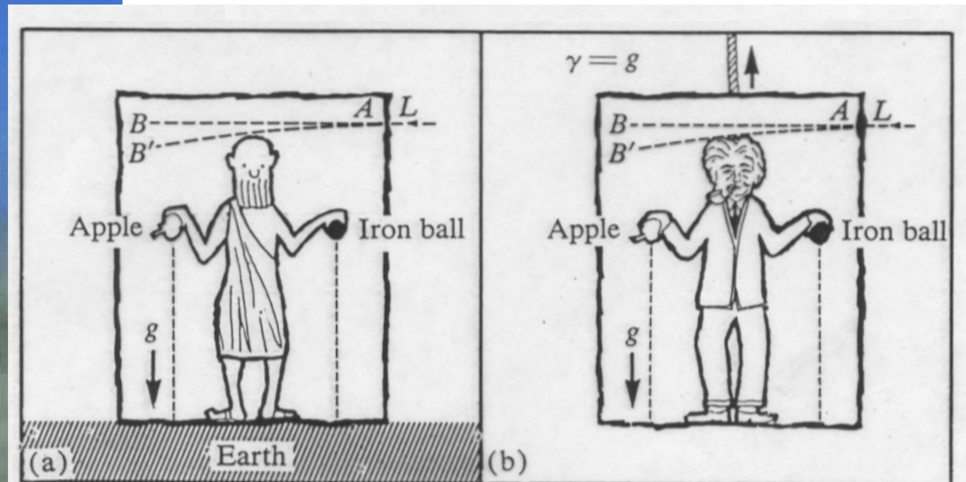
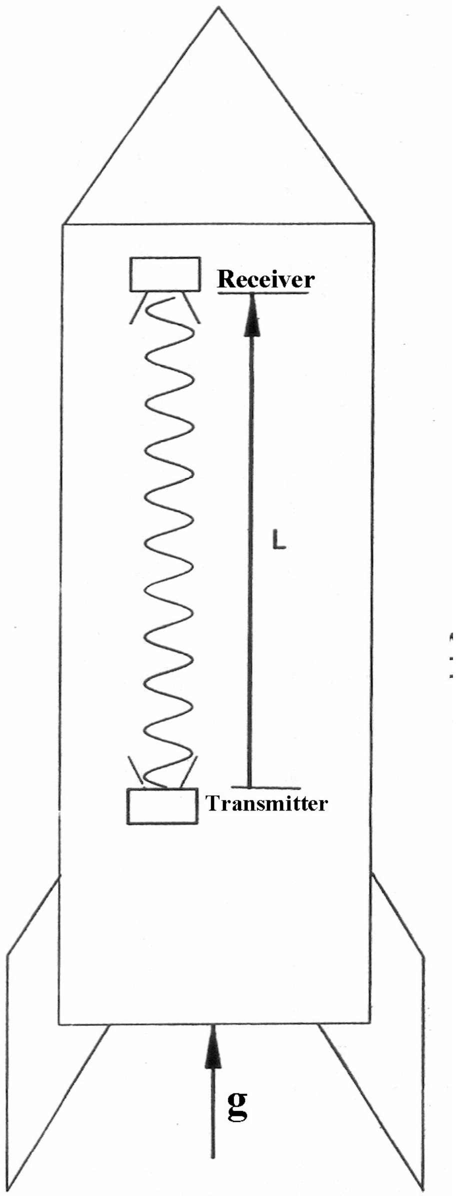


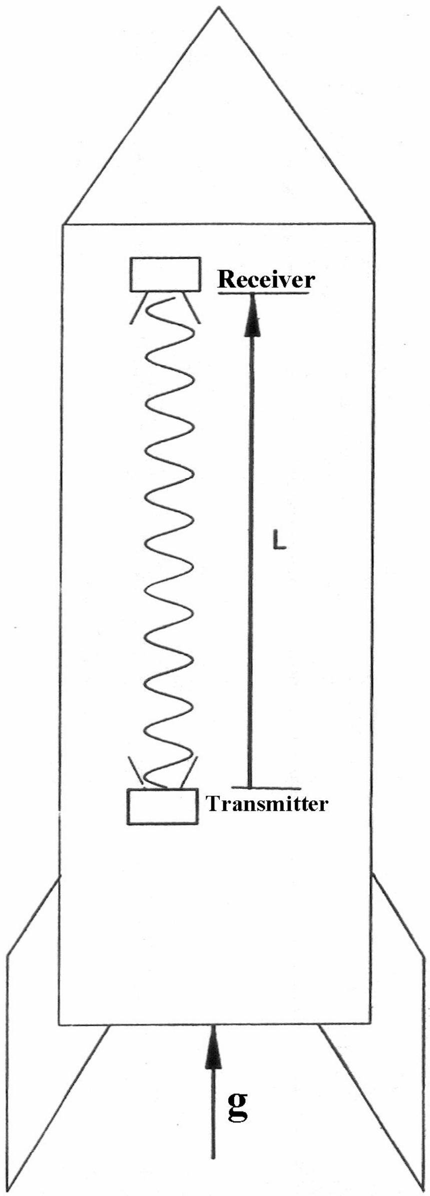
Diagram from H. Yilmaz, "Introduction to Theory of Relativity and Principles of Modern Physics," Blaisdell, NY (1965).

Over a small region of space and time, a fictitious gravity field induced by acceleration cannot be distinguished From a gravity field produced by mass.

Gravitational Frequency Shift



Gravitational Frequency Shift



$$t = L/c$$

$$v = gt = \frac{gL}{c};$$

$$\frac{\Delta f}{f} = -\frac{v}{c} = -\frac{gL}{c^2} = -\frac{\Delta\Phi}{c^2}.$$

$$\frac{\Delta f}{f} = -\frac{-\frac{GM}{r} - \left(-\frac{GM}{a_1}(1+\frac{J_2}{2})\right)}{c^2}$$

The situation in the rocket is static. The fractional frequency difference between the clocks is

$$\frac{\Delta f}{f} = +\frac{\Delta\Phi}{c^2}.$$

Frequency shifts due to Gravitational Potential Differences

Let Φ_0 be the gravitational potential on earth's geoid--at mean sea level, and r be the radius of a GPS satellite.

$$\Delta\Phi = -\frac{GM_E}{r} - \Phi_0$$

Φ_0 includes contributions from earth's oblateness as well as its mass:

$$\Phi_0 = -\frac{GM}{a_1}(1 + \frac{1}{2}J_2),$$

where: a_1 is the equatorial radius of the earth;

$J_2 \simeq .001086$ is earth's quadrupole moment coefficient.

How big are gravitational frequency shifts in the GPS?

To get a rough estimate, assume the satellite orbit is circular and the reference clock is on earth's equator.

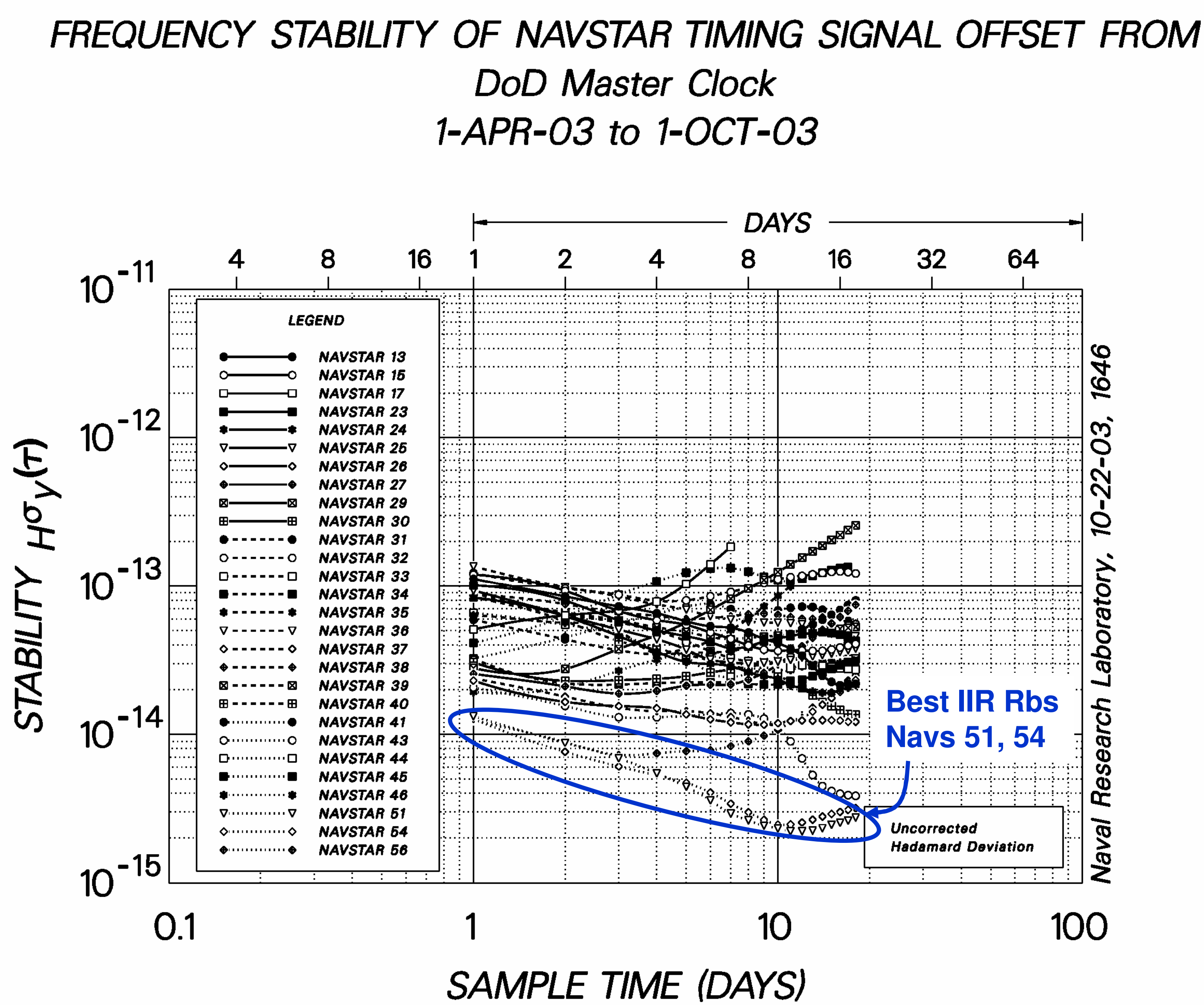
$$\frac{\Delta f}{f} = \frac{1}{c^2}\left(-\frac{GM}{a} - \left(-\frac{GM}{a_1}\right)\right)$$

where $$GM = 3.986\times10^{14}\,\mathrm{m^3/s^2};$$

$$a = 26{,}562\,\mathrm{km};$$

$$a_1 = 6{,}378\,\mathrm{km}$$

$$\frac{\Delta f}{f} \simeq 5\times10^{-10};\ \ (\approx 13\,\mathrm{km\ navigation\ error\ per\ day})$$



FREQUENCY STABILITY OF NAVSTAR TIMING SIGNAL OFFSET FROM
DoD Master Clock
1-APR-03 to 1-OCT-03
DAYS
STABILITY $_H\sigma_y(\tau)$
SAMPLE TIME (DAYS)
LEGEND
NAVSTAR 13
NAVSTAR 15
NAVSTAR 17
NAVSTAR 23
NAVSTAR 24
NAVSTAR 25
NAVSTAR 26
NAVSTAR 27
NAVSTAR 29
NAVSTAR 30
NAVSTAR 31
NAVSTAR 32
NAVSTAR 33
NAVSTAR 34
NAVSTAR 35
NAVSTAR 36
NAVSTAR 37
NAVSTAR 38
NAVSTAR 39
NAVSTAR 40
NAVSTAR 41
NAVSTAR 43
NAVSTAR 44
NAVSTAR 45
NAVSTAR 46
NAVSTAR 51
NAVSTAR 54
NAVSTAR 56
Best IIR Rbs
Navs 51, 54
Uncorrected
Hadamard Deviation
Naval Research Laboratory, 10-22-03, 1646

Question:

If a clock makes an error in one day of 1 part in 10^14, how far would light travel in this amount of time?

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Answer:

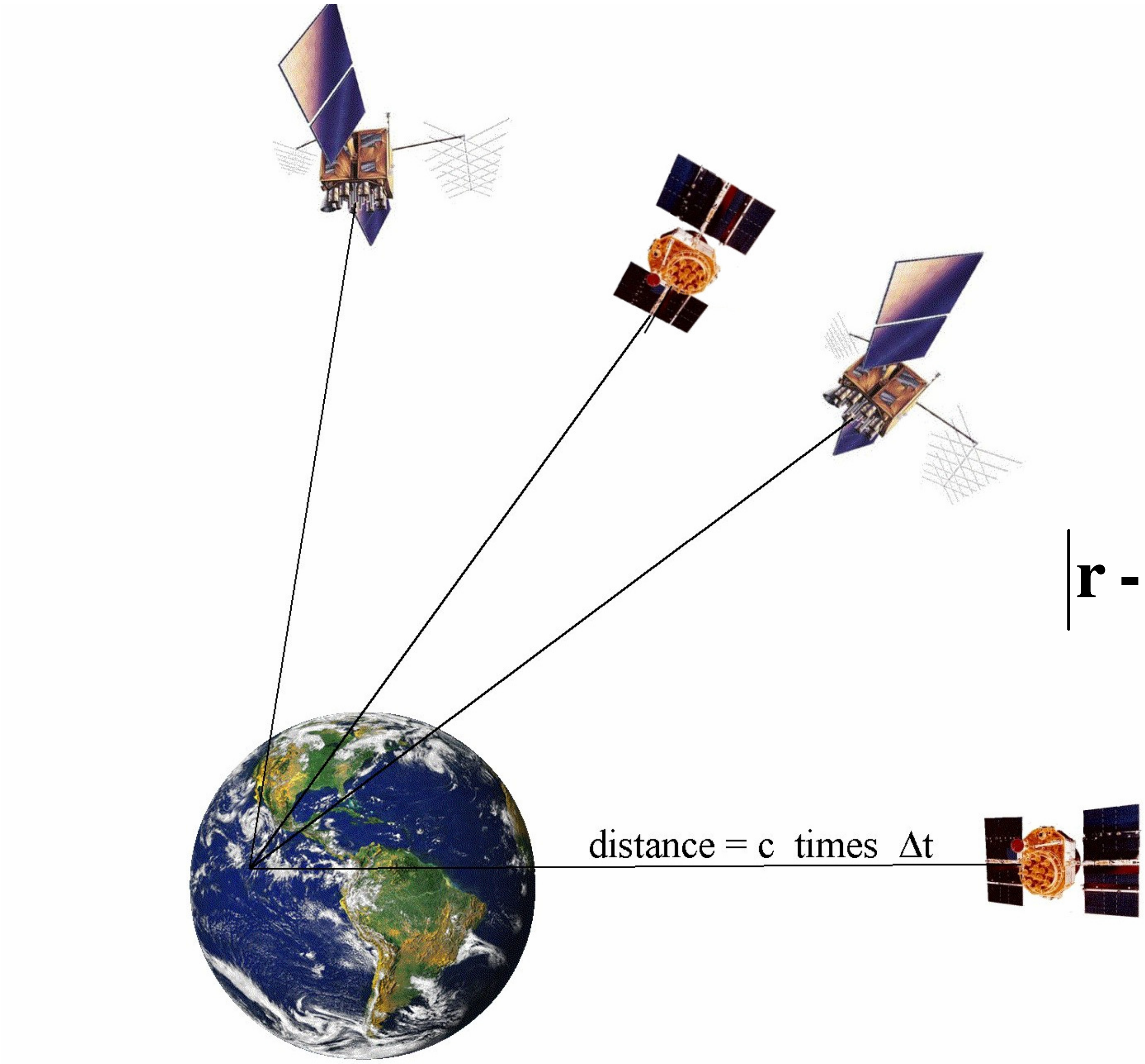
In one day, the error is:

$$10^{-14} \times 86400\,\text{sec} = 8.64 \times 10^{-10}\,\text{sec}.$$

In this amount of time, light travels a distance

$$d = c \times (8.64 \times 10^{-10}\,\text{sec}) = 299792458\text{m/sec} \times (8.64 \times 10^{-10}\,\text{sec})$$
$$= 0.26\ \text{m}$$

Constancy of c



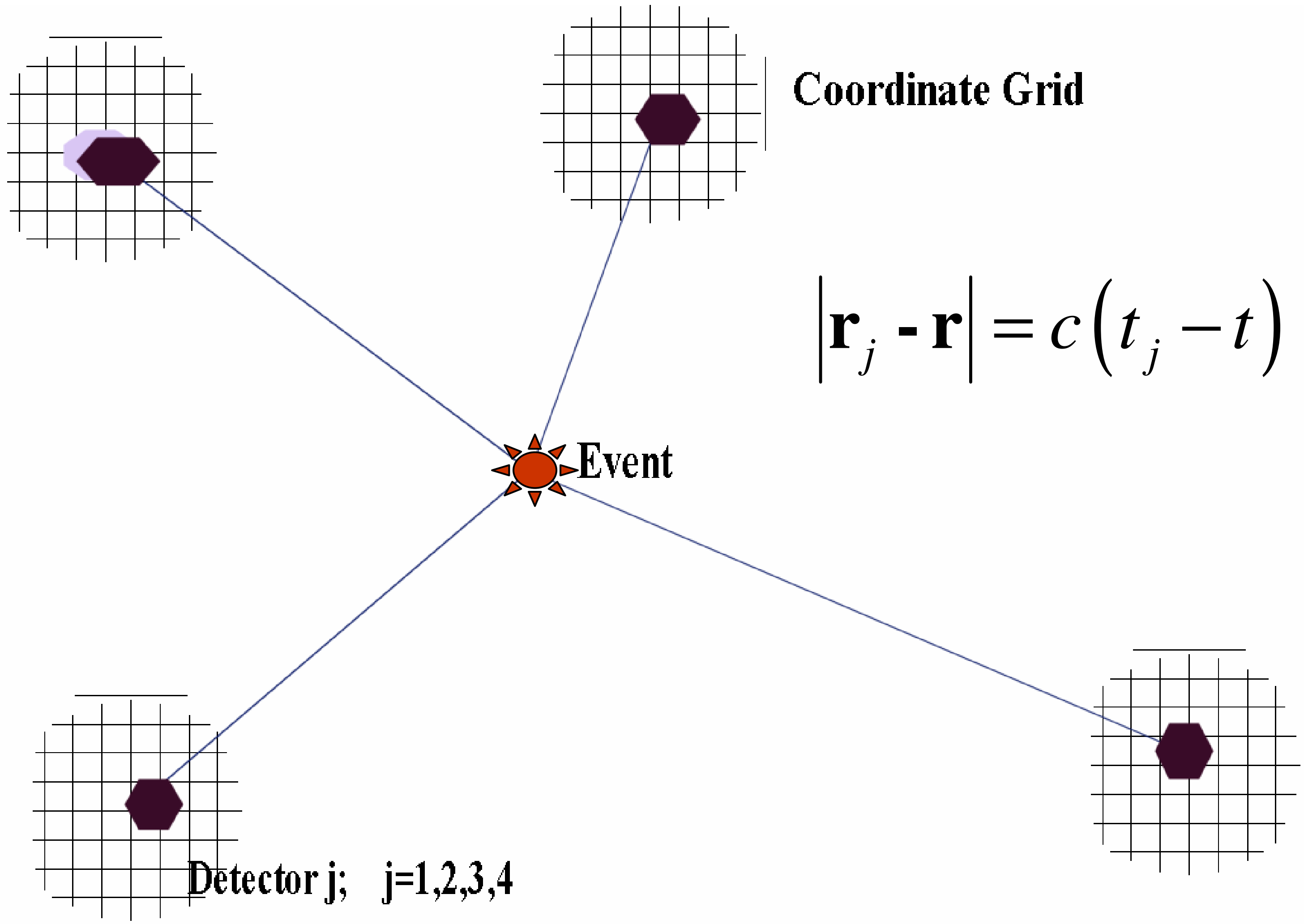
$$\left|\mathbf{r} - \mathbf{r}_j\right| = c\left(t - t_j\right),$$
$$j = 1, 2, 3, 4$$

Sample data

SV #	Transmission Epoch t_i (s)	Transmitter Position x_i (m)	Transmitter Position y_i (m)	Transmitter Position z_i (m)
1	37239. 924 422 365 6	13004579.642	18997823.895	13246718.721
2	37239. 920 713 391 8	20450127.566	16360459.358	-4436309.875
3	37239. 925 307 870 0	20982631.270	15908390.245	3486595.546
4	37239. 929 346 353 9	13799439.294	-8705178.668	20959777.407

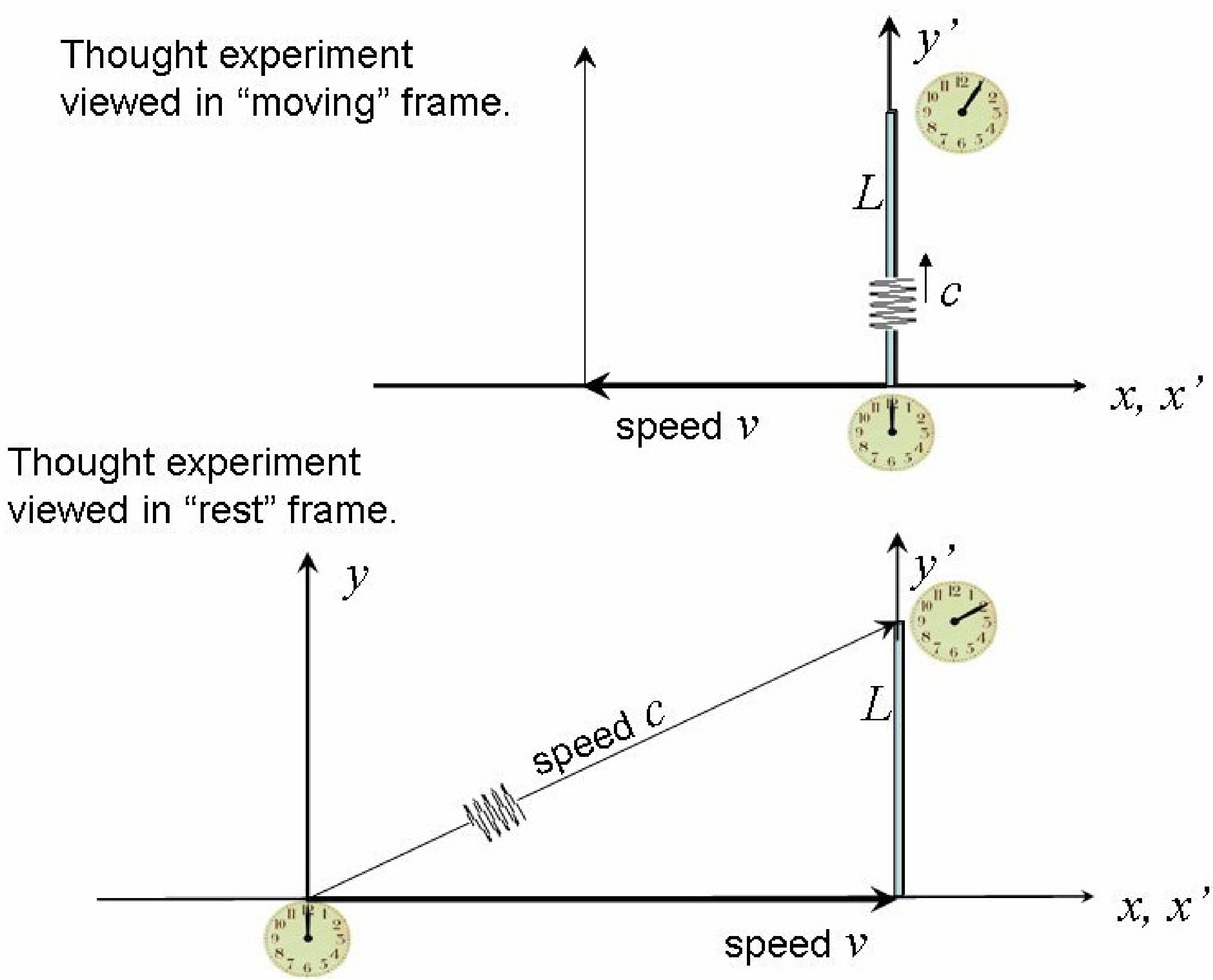
Question: Where is the receiver and what is the time at the receiver?

Reciprocity

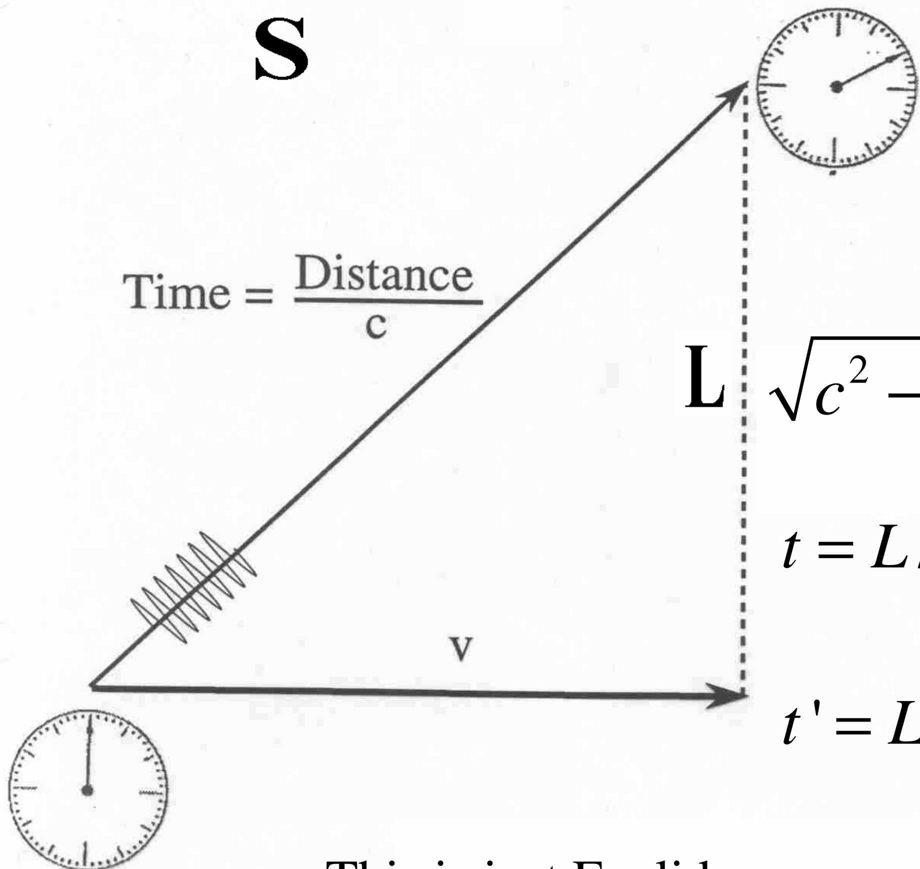


$$\left|\mathbf{r}_j - \mathbf{r}\right| = c\left(t_j - t\right)$$

The constancy of the speed of light implies time dilation



Einstein’s Light Clock



$$\sqrt{c^2 - v^2}$$

$$t = L / \sqrt{c^2 - v^2} = \frac{L/c}{\sqrt{1 - v^2/c^2}};$$

$$t' = L/c = \sqrt{1 - v^2/c^2}\; t$$

This is just Euclidean geometry and the constancy of c.

How Big is Time Dilation in the GPS?

$$\sqrt{1 - v^2 / c^2} \approx 1 - \frac{1}{2}\frac{v^2}{c^2};$$

$$v = 4000\,\mathrm{m/s};$$

$$-\frac{1}{2}\frac{v^2}{c^2} = -3.5 \times 10^{-11}$$

Accounting For Relativistic Effects

Example: Time Dilation:

$$d\tau = \sqrt{1 - v^2 / c^2}\, dt;$$

$$dt = \left(1 - v^2 / c^2\right)^{-1/2} d\tau$$

$$\simeq \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) d\tau.$$

Elapsed Coordinate time:

$$\Delta t = \int_{\text{path}} d\tau \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)$$

Observed Proper Time

GPS Satellite in a circular earth-bound orbit

Newton's law of motion: Force toward the earth is gravitational.

$$-\frac{GmM}{r^3}\mathbf{r} = m\ddot{\mathbf{r}}\,;$$

The mass of the satellite cancels out (a consequence of the Principle of Equivalence). The orbit is nearly a Kepler ellipse. Solution of the above equation shows that the energy is constant:

$$\frac{1}{2}v^2 - \frac{GM}{r} = -\frac{GM}{2a}$$

where a is the semimajor axis.

Frequency shift of GPS satellite clocks relative to reference clock on equator

$$\frac{\Delta f}{f} = \frac{\Phi - \Phi_0}{c^2} - \frac{1}{2}\frac{v^2}{c^2} - \left(-\frac{1}{2}\frac{(\omega_E a_1)^2}{c^2} \right)$$

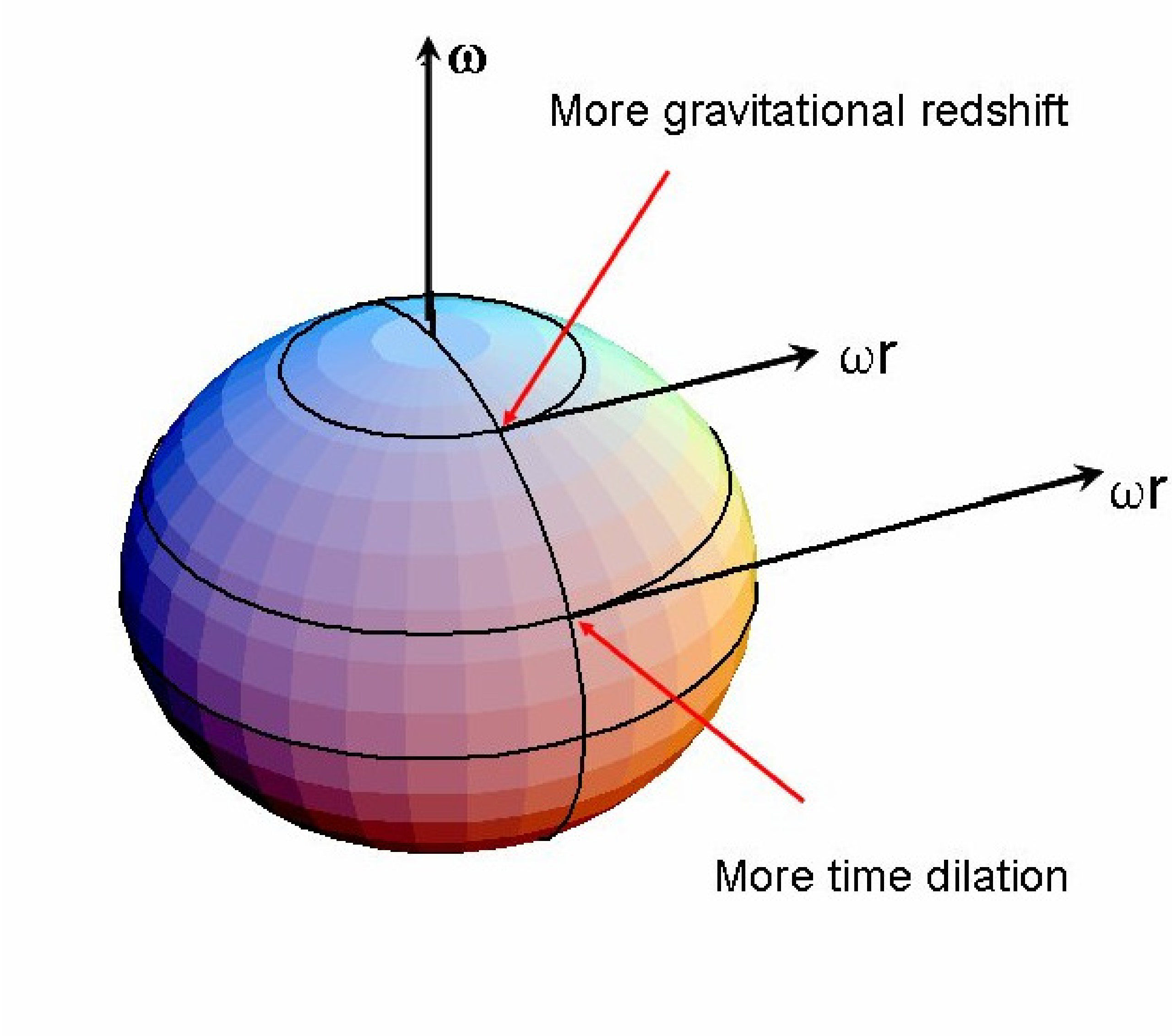
Putting in everything that is known about these quantities and adding and subtracting some terms,

$$\frac{\Delta f}{f} = \frac{1}{c^2}\left(-2GM_E\left(\frac{1}{a} - \frac{1}{r}\right) - \frac{3GM_E}{2a} + \frac{GM_E}{a_1}(1 + J_2/2) + \frac{1}{2}(\omega_E a_1)^2 \right)$$
$$= -\frac{2GM_E}{c^2}\left(\frac{1}{a} - \frac{1}{r}\right) + 4.4647\times 10^{-10}.$$

The first term depends on orbital eccentricity and gives rise to a periodic time error that is (E is the eccentric anomaly)

$$\Delta t_{rel} = -\frac{2e}{c^2}\sqrt{GM_E a}\,\sin(E) + const.$$

Clocks on earth's geoid beat at equal rates



Orbit adjustments in GPS

$$\Delta \frac{\delta f}{f} = \Delta\left(-\frac{3GM_E}{2c^2 a} \right) = \frac{3GM_E}{2c^2 a} \frac{\delta a}{a};$$

For an increase in altitude of 20 km, the change in frequency is

$$\Delta \frac{\delta f}{f} \simeq 1.88 \times 10^{-13}.$$

This effect is now understood and the clock frequency is adjusted when the orbit is changed. (Back-of-the-envelope calculation!)

Further development--introduce the metric

$$ds^2 = -(1 + \frac{2\Phi}{c^2})(c\,dt)^2 + dx^2 + dy^2 + dz^2;$$

Discuss proper time, coordinate time; show how the effects can be obtained from the metric…..

END

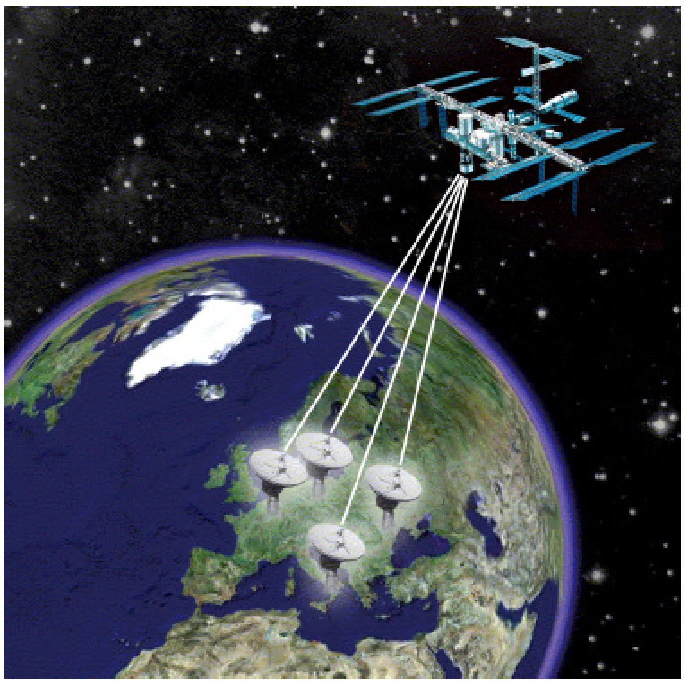
Sagnac effect

Common View

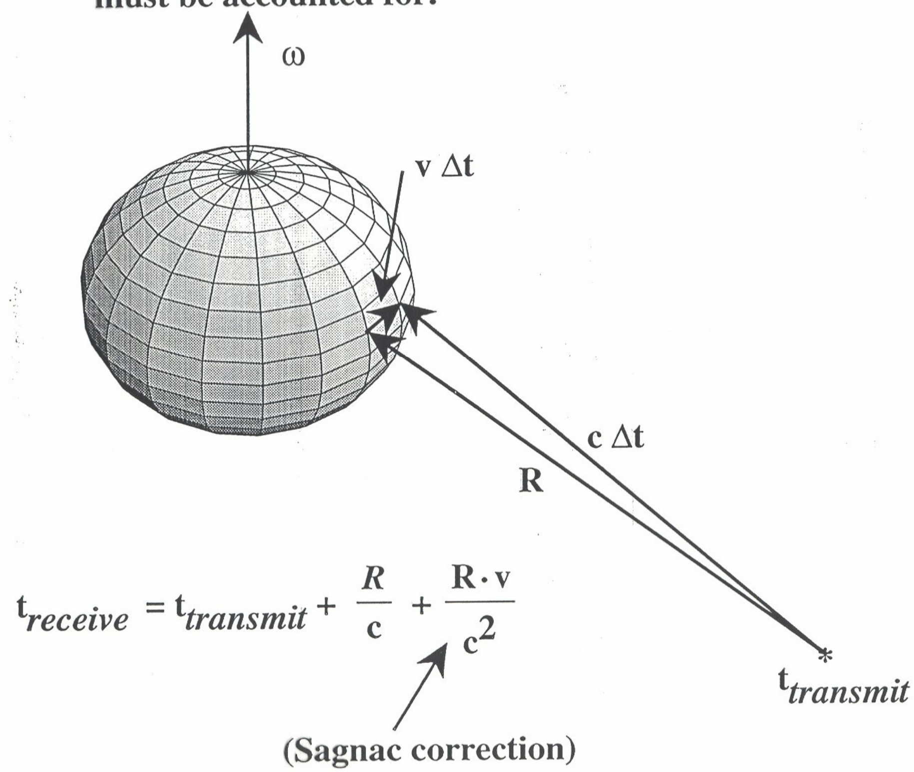
Ground to ground clock comparison

Governed by short-term stability

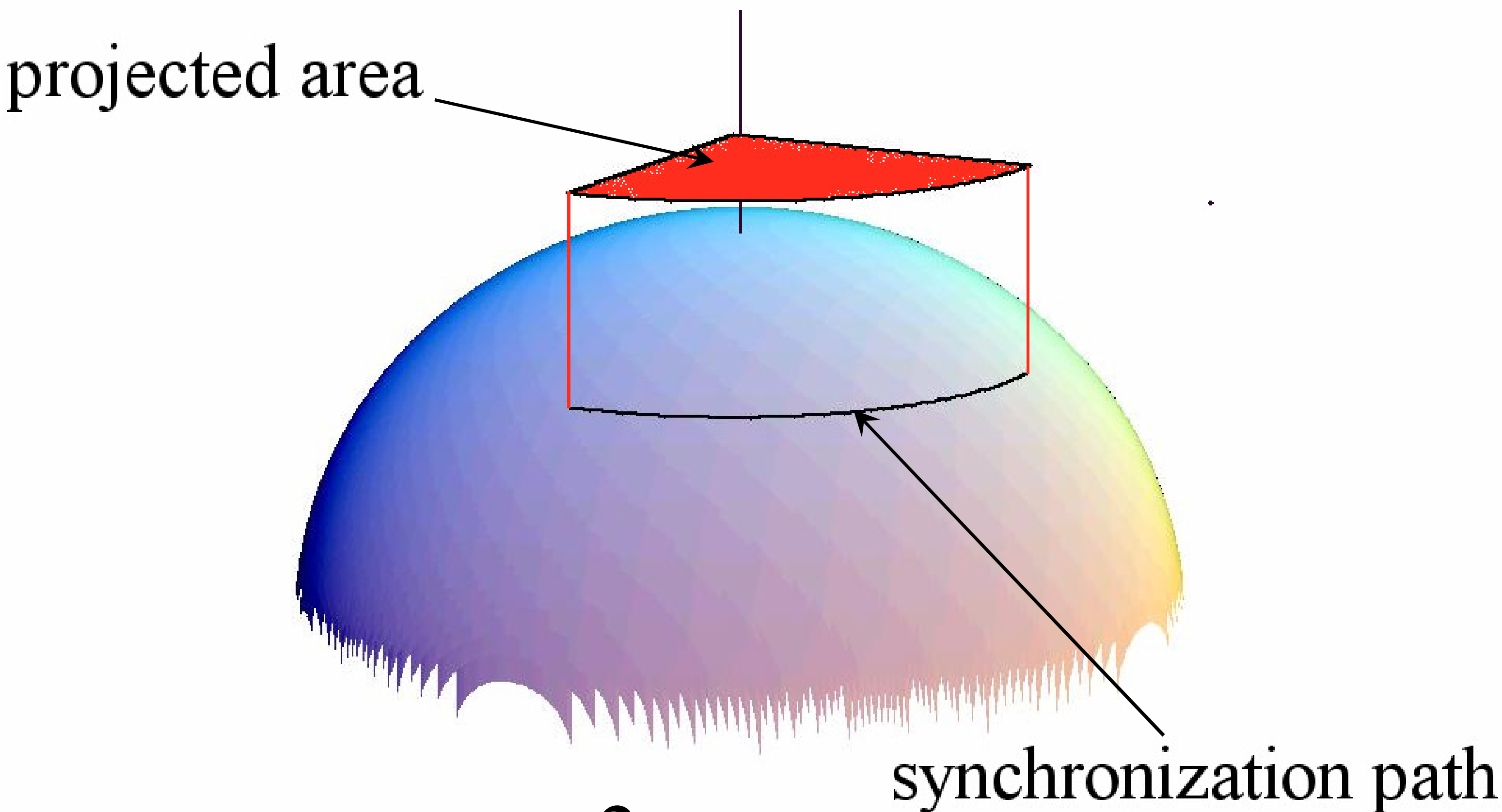
< 0.3-ps over one ISS pass (300 s)



Receiver motion in ECI frame during signal propagation must be accounted for.

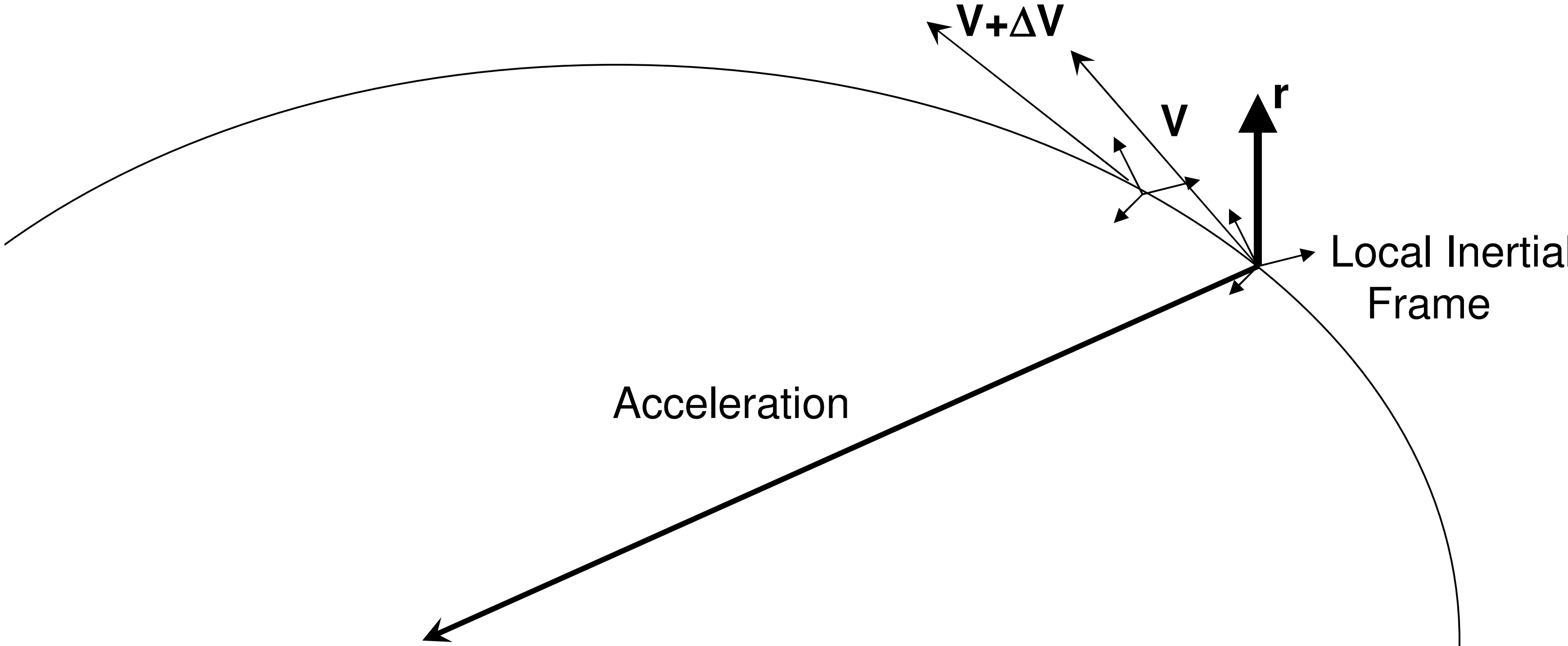


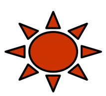
Sagnac Effect on Synchronization in a Rotating System



Sagnac Correction = $$\frac{2\omega}{c^2} A_z$$

Principle of Equivalence



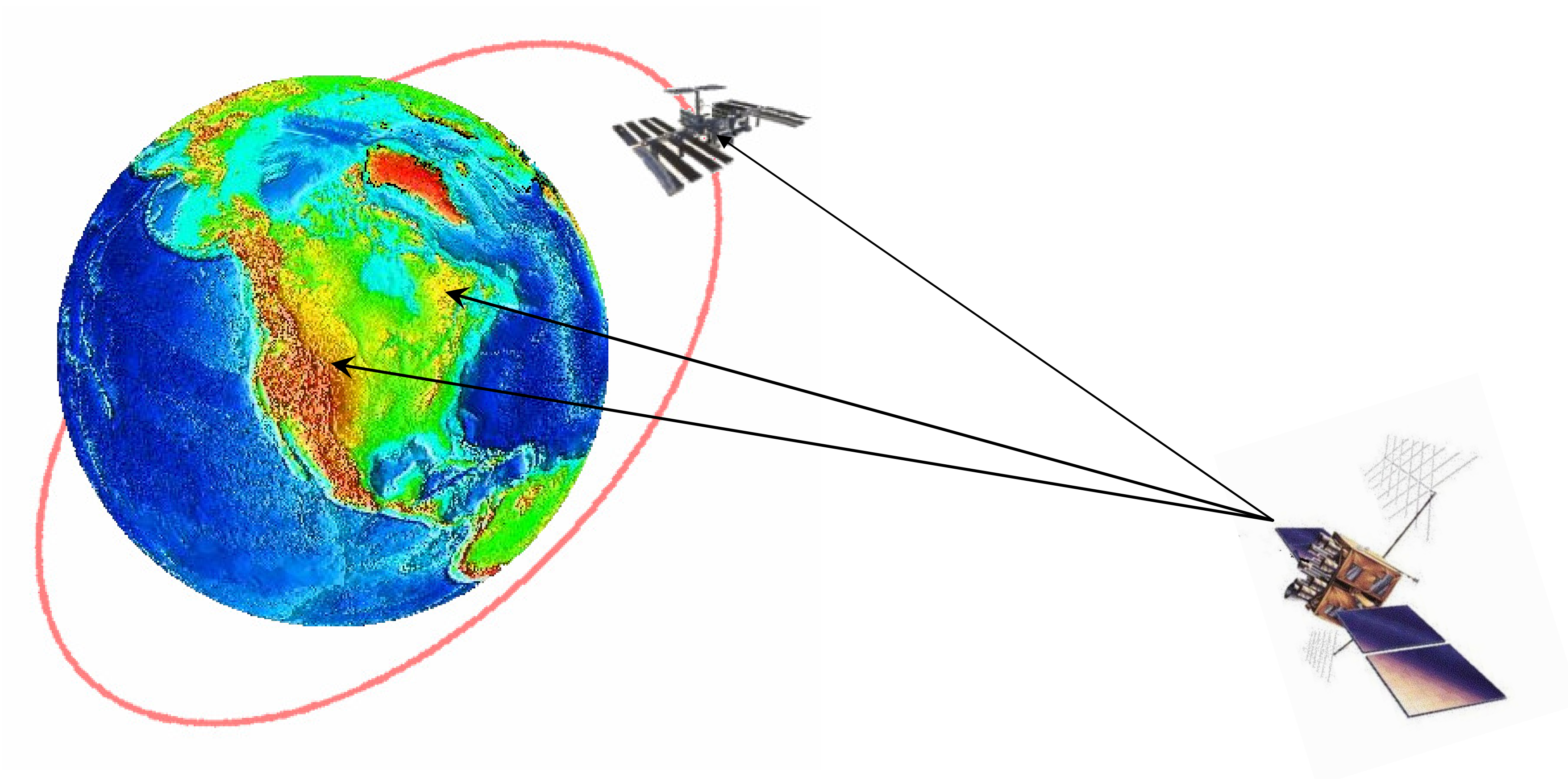


Induced potential difference/c^2 $= \dfrac{\mathbf{A}\bullet\mathbf{r}}{c^2} = \dfrac{\Delta\mathbf{V}\bullet\mathbf{r}}{\Delta T c^2}$;

Gravitational potential difference/c^2 $= \dfrac{\nabla\Phi\bullet\mathbf{r}}{c^2}$;

Net potential difference/c^2 $= \dfrac{\mathbf{A}\bullet\mathbf{r}+\nabla\Phi\bullet\mathbf{r}}{c^2} = \dfrac{1}{\Delta T}\dfrac{\Delta\mathbf{V}\bullet\mathbf{r}}{c^2} + \dfrac{\nabla\Phi\bullet\mathbf{r}}{c^2} = 0$;

Common-view time transfer



Why are atomic clocks needed?

To reduce the effect of clock error to < 2 meters,

the clock error must be less than $2/c = 6.7 \times 10^{-9}$ sec.

Half a day = 43200 seconds, so the fractional clock error must be less than:

$$(2\ \text{m})/(43200\ \text{s} \times c) = \mathbf{1.5 \times 10^{-13}}.$$

Only atomic clocks can achieve such stability.