AAPT UNITED STATES PHYSICS TEAM
AIP CEE 2013

2013 $F = ma$ Contest

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet.
- Your answer to each question must be marked on the optical mark answer sheet.
- Select the single answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased.
- Correct answers will be awarded one point; incorrect answers will result in a deduction of $\frac{1}{4}$ point. There is no penalty for leaving an answer blank.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- This test contains 25 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily the same level of difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 20, 2013.**
- The question booklet and answer sheet will be collected at the end of this exam. You may not use scratch paper.

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Contributors to this year's exam include David Fallest, Jiajia Dong, Paul Stanley, Warren Turner, Qiuzi Li, and former US Team members Marianna Mao, Andrew Lin, Steve Byrnes.

1. An observer stands on the side of the front of a stationary train. When the train starts moving with constant acceleration, it takes 5 seconds for the first car to pass the observer. How long will it take for the 10th car to pass?

   (A) 1.07s
   (B) 0.98s
   (C) 0.91s
   (D) 0.86s
   (E) 0.81s

Solution

Start with

$$\Delta x = \frac{1}{2}at^2 + v_i t$$

We have four times. $t_0 = 0$ is when the train starts, and when the first car is aligned with the observer. t_1 is when the end of the first car is aligned with the observer. Then

$$L = \frac{1}{2}a{t_1}^2$$

We are assuming the car has a length L. t_2 is when the tenth car is first aligned with the observer, so

$$9L = \frac{1}{2}a{t_2}^2$$

and finally, t_3 is when that car has passed,

$$10L = \frac{1}{2}a{t_3}^2$$

From the equation for t_1 we find

$$2L/a = 25 \text{ s}^2$$

so

$$t_2 = \sqrt{9 \cdot 25 \text{ s}^2} = 15 \text{ s}$$

and

$$t_3 = \sqrt{10 \cdot 25 \text{ s}^2} = 15.81 \text{ s}$$

2. Jordi stands 20 m from a wall and Diego stands 10 m from the same wall. Jordi throws a ball at an angle of 30° above the horizontal, and it collides elastically with the wall. How fast does Jordi need to throw the ball so that Diego will catch it? Consider Jordi and Diego to be the same height, and both are on the same perpendicular line from the wall.

   (A) 11 m/s
   (B) 15 m/s
   (C) 19 m/s
   (D) 30 m/s
   (E) 35 m/s

Solution

The wall acts like a mirror for a perfectly elastic collision. In that case, Jordi and Diego are effectively 30 meters apart. Using the range formula,

$$R = \frac{v^2}{g} \sin 2\theta,$$

we get

$$v^2 = \frac{gR}{\sin 2\theta} \approx \frac{300 \text{ m}^2/\text{s}^2}{\sqrt{3}/2}$$

$$v \approx 19 \text{ m/s}$$

3. Tom throws a football to Wes, who is a distance l away. Tom can control the time of flight t of the ball by choosing any speed up to v_{max} and any launch angle between 0° and 90°. Ignore air resistance and assume Tom and Wes are at the same height. Which of the following statements is **incorrect**?

   (A) If $v_{max} < \sqrt{gl}$, the ball cannot reach Wes at all.

   (B) Assuming the ball can reach Wes, as v_{max} increases with l held fixed, the minimum value of t decreases.

   (C) Assuming the ball can reach Wes, as v_{max} increases with l held fixed, the maximum value of t increases.

   (D) Assuming the ball can reach Wes, as l increases with v_{max} held fixed, the minimum value of t increases.

   (E) Assuming the ball can reach Wes, as l increases with v_{max} held fixed, the maximum value of t increases.

Solution

The greater the initial vertical velocity of the football, the longer it will take to fall back to the ground. Meanwhile, the initial horizontal velocity of the football increases with l and decreases with the time of flight.

Thus the minimum time of flight is obtained by the minimum possible initial vertical velocity; it is limited by the increasing required horizontal velocity, and this limitation becomes more severe as l increases. The maximum time of flight is obtained by the maximum possible initial vertical velocity; it is limited by the required horizontal velocity component, and this limitation likewise becomes more severe as l increases. Thus increasing v_{max} expands the range of available time of flight in both directions; increasing l contracts it in both directions.

(A) follows from the standard result for the maximum range of a projectile.

For those who prefer the mathematical approach, we have for a launch angle θ and initial speed v_0

$$2v_0 \sin\theta = gt$$

$$(v_0 \cos\theta)t = l$$

Since $\sin^2\theta + \cos^2\theta = 1$,

$${v_0}^2 = \left(\frac{gt}{2}\right)^2 + \left(\frac{l}{t}\right)^2$$

$$\frac{g^2}{4}t^4 - {v_0}^2 t^2 + l^2 = 0$$

$$t^2 = 2\frac{{v_0}^2 \pm \sqrt{{v_0}^4 - g^2 l^2}}{g^2}$$

(A) follows immediately from the requirement that the result be real. It is also clear that increasing l contracts the range of available times in both directions, and that increasing v_{max} increases the maximum available time. As for the minimum available time, we have

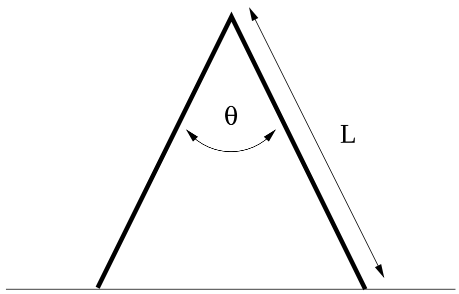
$${t_-}^2 = 2\frac{{v_0}^2 - \sqrt{{v_0}^4 - g^2 l^2}}{g^2}$$

$$\frac{\partial\left({t_-}^2\right)}{\partial\left({v_0}^2\right)} = \frac{2}{g^2}\left(1 - \frac{2{v_0}^2}{2\sqrt{{v_0}^4 - g^2 l^2}}\right)$$

$$\frac{\partial\left({t_-}^2\right)}{\partial\left({v_0}^2\right)} = \frac{2}{g^2}\left(1 - \frac{1}{\sqrt{1 - \frac{g^2 l^2}{{v_0}^4}}}\right)$$

The right hand side is always negative, so t_- is always minimized by choosing the largest possible v_0.

4. The sign shown below consists of two uniform legs attached by a frictionless hinge. The coefficient of friction between the ground and the legs is μ. Which of the following gives the maximum value of θ such that the sign will not collapse?



(A) $\sin\theta = 2\mu$
(B) $\sin\theta/2 = \mu/2$
(C) $\tan\theta/2 = \mu$
(D) $\tan\theta = 2\mu$
(E) $\tan\theta/2 = 2\mu$

Solution

For each leg, friction must balance the horizontal force from the other leg at the top. Balancing torques,

$$\frac{L}{2}(M/2)g\sin\alpha = LF_f\cos\alpha$$

where F_f is the force of friction on the leg, and $\alpha = \frac{1}{2}\theta$.
The upward normal force on the leg is equal to the weight of the leg, so

$$F_N = (M/2)g$$

Equating,

$$\mu F_N = F_f$$

or

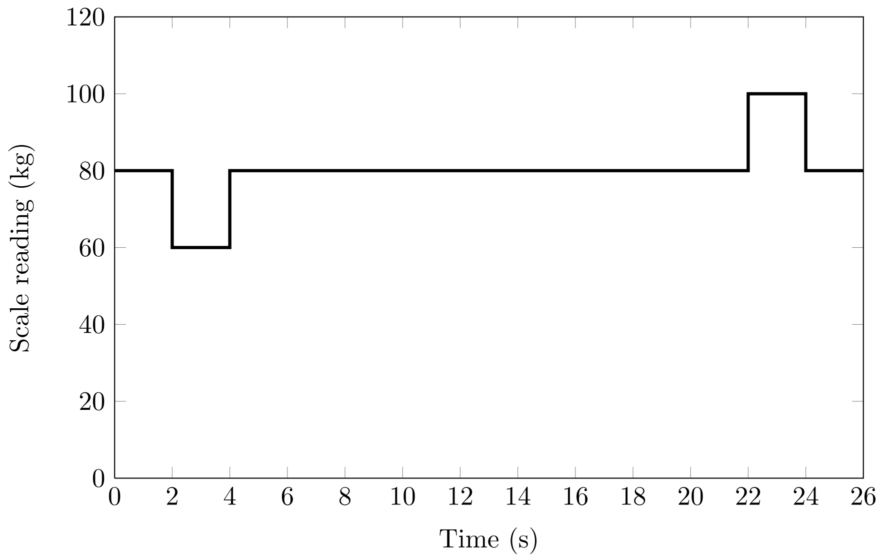
$$\frac{L}{2}(M/2)g\sin\alpha = L\mu(M/2)g\cos\alpha$$

or

$$\tan\theta/2 = 2\mu$$

The following information applies to questions 5 and 6

A student steps onto a stationary elevator and stands on a bathroom scale. The elevator then travels from the top of the building to the bottom. The student records the reading on the scale as a function of time.



5. At what time(s) does the student have maximum downward velocity?

    (A) At all times between 2 s and 4 s
    (B) At 4 s only
    (C) At all times between 4 s and 22 s
    (D) At 22 s only
    (E) At all times between 22 s and 24 s

6. How tall is the building?

    (A) 50 m
    (B) 80 m
    (C) 100 m
    (D) 150 m
    (E) 400 m

Solution

The bathroom scale does not directly measure the weight of the student; instead it measures the normal force F_N supporting her feet, scaled by g.

$$m_{scale}g = F_N$$

The normal force acts upward on the student. Gravity, of course, always exerts a force mg downward. Thus her downward acceleration a is given by

$$ma = mg - F_N$$

Combining these and rearranging,

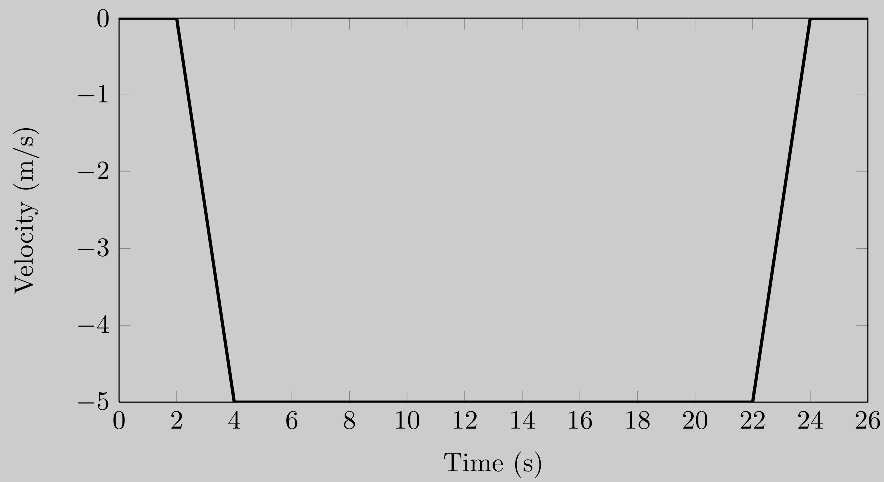
$$a = g\left(1 - \frac{m_{scale}}{m}\right)$$

Since the elevator begins at rest, the student's acceleration is initially zero, so her mass must be 80 kg. From the graph we see that there are two periods of uniform acceleration; the first is downward with magnitude

$$a = (10\ \text{m/s}^2)\left(1 - \frac{60\ \text{kg}}{80\ \text{kg}}\right) = 2.5\ \text{m/s}^2$$

and the second is upward with the same magnitude. The maximum downward velocity occurs between the periods of uniform acceleration, when the car travels at a constant speed; this occurs between 4 s and 22 s.

The downward acceleration lasts for 2 s, so the speed of the elevator after the acceleration is 5 m/s. The period between accelerations is approximately 20 s, and so the distance traveled is approximately 100 m. In fact, this calculation yields the exact answer, as can be seen by considering the exact graph of velocity *vs.* time:



The displacement is the area between the curve and the horizontal axis, which is exactly 100 m.

7. A light car and a heavy truck have the same momentum. The truck weighs ten times as much as the car. How do their kinetic energies compare?

(A) The truck's kinetic energy is larger by a factor of 100

(B) They truck's kinetic energy is larger by a factor of 10

(C) They have the same kinetic energy

(D) The car's kinetic energy is larger by a factor of 10

(E) The car's kinetic energy is larger by a factor of 100

Solution

Express kinetic energy in terms of momentum,

$$K = \frac{p^2}{2m}$$

Then the ratio is

$$\frac{K_1}{K_2} = \frac{m_2}{m_1}$$

The following information applies to questions 8 and 9

A truck is initially moving at velocity v. The driver presses the brake in order to slow the truck to a stop. The brake applies a constant force F to the truck. The truck rolls a distance x before coming to a stop, and the time it takes to stop is t.

8. Which of the following expressions is equal the initial kinetic energy of the truck (i.e. the kinetic energy before the driver starts braking)?

   (A) Fx
   (B) Fvt
   (C) Fxt
   (D) Ft
   (E) Both (a) and (b) are correct

Solution

Considering magnitudes only,

$$K_i = \delta K = W = Fx$$

9. Which of the following expressions is equal the initial momentum of the truck (i.e. the momentum before the driver starts braking)?

   (A) Fx
   (B) $Ft/2$
   (C) Fxt
   (D) $2Ft$
   (E) $2Fx/v$

Solution

Considering magnitudes only,

$$p_i = \Delta p = Ft$$

but

$$x = \frac{1}{2}at^2 = \frac{1}{2}(at)t = \frac{1}{2}vt$$

so

$$p_i = 2Fx/v$$

10. Which of the following can be used to distinguish a solid ball from a hollow sphere of the same radius and mass?

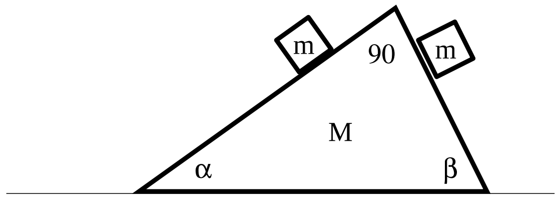
(A) Measurements of the orbit of a test mass around the object.
(B) Measurements of the time it takes the object to roll down an inclined plane.
(C) Measurements of the tidal forces applied by the object to a liquid body.
(D) Measurements of the behavior of the object as it floats in water.
(E) Measurements of the force applied to the object by a uniform gravitational field.

Solution

The measurement described in (B) is certainly appropriate; the solid ball has a smaller moment of inertia than the hollow sphere and will accelerate down the inclined plane faster.

The measurements in (A) and (C) are unhelpful because they only probe the gravitational field outside the object; for a spherically symmetric object this depends only on the mass. The force measured in (E) depends only on the object's total mass; the buoyant force applied in (D) depends only on the external shape of the object.

11. A right-triangular wooden block of mass M is at rest on a table, as shown in figure. Two smaller wooden cubes, both with mass m, initially rest on the two sides of the larger block. As all contact surfaces are frictionless, the smaller cubes start sliding down the larger block while the block remains at rest. What is the normal force from the system to the table?



(A) $2mg$

(B) $2mg + Mg$

(C) $mg + Mg$

(D) $Mg + mg(\sin\alpha + \sin\beta)$

(E) $Mg + mg(\cos\alpha + \cos\beta)$

Solution

Two forces act on each cube: the normal force from the triangular block, and gravity. The normal force must balance the normal component of gravity, which in the case of the left cube is

$$F_N = mg\cos\alpha$$

The vertical component of this normal force is transmitted through the triangular block to the ground, and is

$$F_{Ny} = mg\cos^2\alpha$$

A similar result holds for the other cube, and in addition the ground must support the weight of the triangular block; thus in total

$$F_{tot} = mg\cos^2\alpha + mg\cos^2\beta + Mg$$

However, because the block is a right triangle, $\cos^2\alpha + \cos^2\beta = 1$, so that

$$F_{tot} = mg + Mg$$

Note that the horizontal component of the normal force due to the left cube is

$$F_{Nx} = mg\cos\alpha\sin\alpha$$

The right cube likewise applies a normal force with horizontal component $mg\cos\beta\sin\beta$ in the other direction. But, once again, because the block is a right triangle, $\cos\alpha\sin\alpha = \cos\beta\sin\beta$, so the net horizontal force is zero! This justifies the given assumption that the triangular block does not slide on the table.

12. A spherical shell of mass M and radius R is completely filled with a frictionless fluid, also of mass M. It is released from rest, and then it rolls without slipping down an incline that makes an angle θ with the horizontal. What will be the acceleration of the shell down the incline just after it is released? Assume the acceleration of free fall is g.

The moment of inertia of a thin shell of radius r and mass m about the center of mass is $I = \frac{2}{3}mr^2$; the moment of inertia of a solid sphere of radius r and mass m about the center of mass is $I = \frac{2}{5}mr^2$.

(A) $a = g\sin\theta$
(B) $a = \frac{3}{4}g\sin\theta$
(C) $a = \frac{1}{2}g\sin\theta$
(D) $a = \frac{3}{8}g\sin\theta$
(E) $a = \frac{3}{5}g\sin\theta$

Solution

One can use torque or energy to solve this problem.

The torque about an axis through the point of contact is

$$\tau = RF\sin\theta = 2MgR\sin\theta$$

The angular acceleration is given by

$$\tau = I\alpha$$

where the moment of inertia is

$$I = \frac{2}{3}MR^2 + MR^2 + MR^2 = \frac{8}{3}MR^2$$

The acceleration is then

$$a = \alpha R = \frac{2MgR\sin\theta}{\frac{8}{3}MR^2}R = \frac{3}{4}g\sin\theta$$

Alternatively, the kinetic energy of the object is

$$T = \frac{1}{2}(2M)v^2 + \frac{1}{2}\cdot\frac{2}{3}MR^2\omega^2$$

$$T = \frac{4}{3}Mv^2$$

The potential energy is related to the vertical position y by

$$U = -(2M)gy$$

and so by conservation of energy

$$\frac{d}{dt}(T + U) = 0$$

$$\frac{8}{3}Mv\frac{dv}{dt} = -(2M)g\frac{dy}{dt}$$

But

$$\frac{dy}{dt} = -v\sin\theta$$

and so

$$\frac{dv}{dt} = \frac{3}{4}g\sin\theta$$

13. There is a ring outside of Saturn. In order to distinguish if the ring is actually a part of Saturn or is instead part of the satellites of Saturn, we need to know the relation between the velocity v of each layer in the ring and the distance R of the layer to the center of Saturn. Which of the following statements is correct?

    (A) If $v \propto R$, then the layer is part of Saturn.
    (B) If $v^2 \propto R$, then the layer is part of the satellites of Saturn.
    (C) If $v \propto 1/R$, then the layer is part of Saturn.
    (D) If $v^2 \propto 1/R$, then the layer is part of Saturn.
    (E) If $v \propto R^2$, then the layer is part of the satellites of Saturn.

Solution

If attached to Saturn, then $\omega = v/R$ is a constant, so

$$v \propto R$$

If in orbit, then

$$\frac{v^2}{R} = \frac{GM}{r^2},$$

or

$$v^2 \propto 1/R$$

14. A cart of mass m moving at 12 m/s to the right collides elastically with a cart of mass 4.0 kg that is originally at rest. After the collision, the cart of mass m moves to the left with a velocity of 6.0 m/s. Assuming an elastic collision in one dimension only, what is the velocity of the center of mass (v_{cm}) of the two carts *before* the collision?

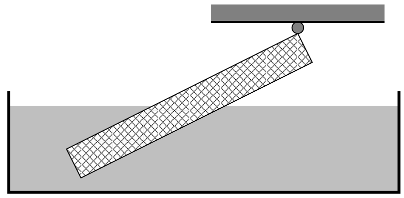
    (A) $v_{\text{cm}} = 2.0$ m/s
    (B) $v_{\text{cm}} = 3.0$ m/s
    (C) $v_{\text{cm}} = 6.0$ m/s
    (D) $v_{\text{cm}} = 9.0$ m/s
    (E) $v_{\text{cm}} = 18$ m/s

Solution

In the center of mass frame, the carts initially have equal and opposite momenta, so that the total momentum is zero. After the elastic collision in this frame the momenta simply reverse direction; this clearly conserves total energy and momentum. The frame of reference in which the velocity of the cart of mass m is reversed travels to the right at 3 m/s, so that the initial velocity is 9 m/s to the right and the final velocity is 9 m/s to the left.

Note that the mass and initial velocity of the other cart are not required at all!

15. A uniform rod is partially in water with one end suspended, as shown in figure. The density of the rod is 5/9 that of water. At equilibrium, what portion of the rod is above water?



(A) 0.25
(B) 0.33
(C) 0.5
(D) 0.67
(E) 0.75

Solution

Suppose a fraction α of the rod is above water. Let the length of the rod be l, the volume of the rod be V, the density of water be ρ_w, and the density of the rod be ρ_r. Consider torques about the pivot point. Gravity applies a torque

$$\tau_g = \rho_r V g \cdot \frac{1}{2} l$$

A fraction $1-\alpha$ of the rod is submerged, and the center of the submerged portion is a distance $(\alpha + \frac{1}{2}(1-\alpha))l$ from the pivot. So the buoyant force applies a torque

$$\tau_b = \rho_w (1-\alpha) V g \cdot \left(\alpha + \frac{1}{2}(1-\alpha)\right) l$$

$$\tau_b = \frac{1}{2}\rho_w (1-\alpha^2) V g l$$

These torques must balance:

$$\tau_g = \tau_b$$

$$\frac{1}{2}\rho_r = \frac{1}{2}\rho_w (1-\alpha^2)$$

$$\frac{\rho_r}{\rho_w} = 1 - \alpha^2$$

We are given $\frac{\rho_r}{\rho_w} = \frac{5}{9}$, so that

$$\frac{5}{9} = 1 - \alpha^2$$

$$\alpha = \frac{2}{3}$$

16. *Inspired by a problem from the 2012 International Physics Olympiad, Estonia.*

A very large number of small particles forms a spherical cloud. Initially they are at rest, have uniform mass density per unit volume ρ_0, and occupy a region of radius r_0. The cloud collapses due to gravitation; the particles do not interact with each other in any other way.

How much time passes until the cloud collapses fully? (The constant 0.5427 is actually $\sqrt{\frac{3\pi}{32}}$.)

(A) $\frac{0.5427}{{r_0}^2\sqrt{G\rho_0}}$

(B) $\frac{0.5427}{r_0\sqrt{G\rho_0}}$

(C) $\frac{0.5427}{\sqrt{r_0}\sqrt{G\rho_0}}$

(D) $\frac{0.5427}{\sqrt{G\rho_0}}$

(E) $\frac{0.5427}{\sqrt{G\rho_0}}r_0$

Solution

This problem is a matter of dimensional analysis. All of the answers have the form

$$t = 0.527G^{-1/2}{\rho_0}^{-1/2}{r_0}^n$$

The dimensions of t are $[T]$, the dimensions of G are $[L]^3[M]^{-1}[T]^{-2}$, the dimensions of ρ_0 are $[L]^{-3}[M]$, and the dimensions of r_0 are $[L]$. So

$$[T] = ([L]^3[M]^{-1}[T]^{-2})^{-1/2}([L]^{-3}[M])^{-1/2}[L]^n$$

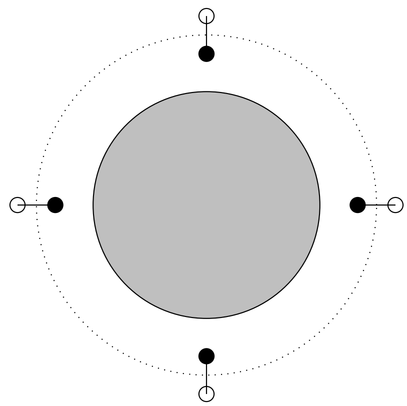
$$[T] = [T][L]^n$$

$$n = 0$$

The time to collapse does not depend on the size of the cloud at all!

17. Two small, equal masses are attached by a lightweight rod. This object orbits a planet; the length of the rod is smaller than the radius of the orbit, but not negligible. The rod rotates about its axis in such a way that it remains vertical with respect to the planet.

- Is there a force in the rod? If so, is it tension or compression?
- Is the equilibrium stable, unstable, or neutral with respect to a small perturbation in the angle of the rod? (Assume this perturbation maintains the rate of rotation, so that in the co-rotating frame the rod is still stationary but at an angle to the vertical.)



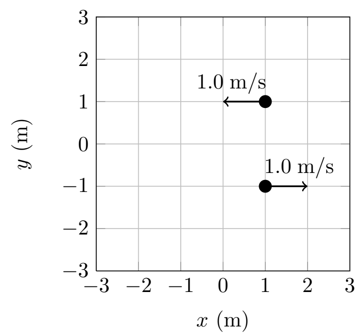
(A) There is no force in the rod; the equilibrium is neutral.
(B) The rod is in tension; the equilibrium is stable.
(C) The rod is in compression; the equilibrium is stable.
(D) The rod is in tension; the equilibrium is unstable.
(E) The rod is in compression; the equilibrium is unstable.

Solution

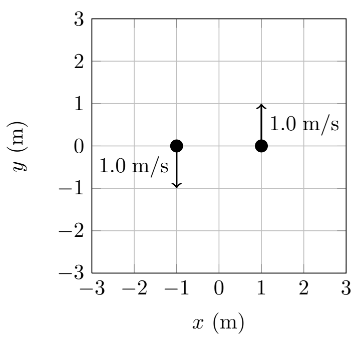
Because the gravitational force goes as $1/r^2$, the inward force on the inner mass is greater than the inward force on the outer mass. On the other hand, both masses have the same angular velocity, so the centripetal acceleration goes as r, and the outer mass must be subject to a greater centripetal force than the inward one. So the rod must exert an inward force on the outer mass and an outward force on the inner mass. Thus the rod is in tension.

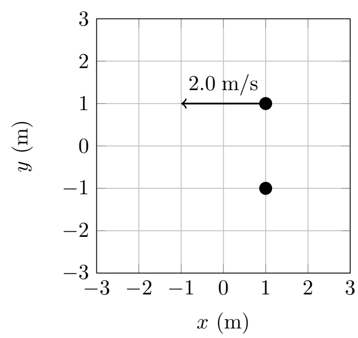
The stability of the position is best understood in the corotating frame. In this frame the outer mass experiences an outward radial force and the inward tension of the rod, and the inner mass experiences an inward radial force and the outward tension of the rod. When the rod is rotated slightly these forces create a restoring torque, so the equilibrium is stable.

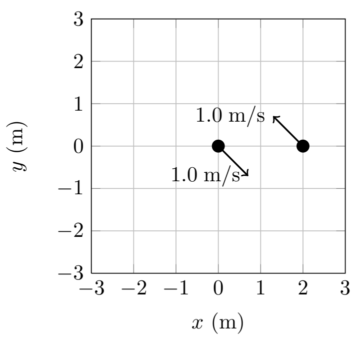
18. Two point particles, each of mass 1 kg, begin in the state shown below.

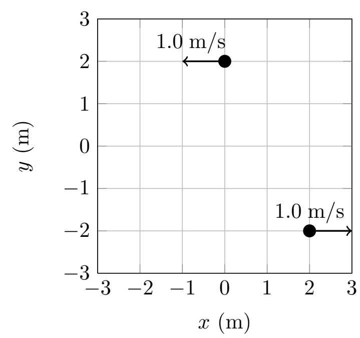


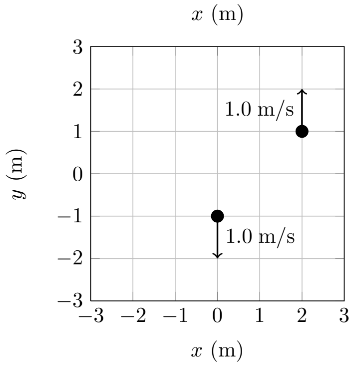
The system evolves through internal forces only. Which of the following could be the state after some time has passed?

(A)


(B)


(C)


(D)


(E)


Solution

First, linear momentum is conserved; as it happens, the system begins with zero linear momentum. This eliminates choice (B).

Angular momentum about any point is also conserved. (It is specified that the particles are point particles to rule out their having spin angular momentum.) As it happens, the system has zero linear momentum, so the angular momentum is the same measured about any point; otherwise it would suffice to consider any one point. The total angular momentum must be 2 kg m^2/s counterclockwise. This eliminates choices (C) and (D).

Finally, because the linear momentum of the system is zero, the center of mass does not move. This eliminates choice (A).

The following information applies to questions 19, 20, and 21.

A simple pendulum experiment is constructed from a point mass m attached to a pivot by a massless rod of length L in a constant gravitational field. The rod is released from an angle $\theta_0 < \pi/2$ at rest and the period of motion is found to be T_0. Ignore air resistance and friction.

19. At what angle θ_g during the swing is the tension in the rod the greatest?

    (A) The tension is the greatest at the point $\theta_g = \theta_0$.
    (B) The tension is the greatest at the point $\theta_g = 0$.
    (C) The tension is the greatest at an angle θ_g with $0 < \theta_g < \theta_0$.
    (D) The tension is constant.
    (E) None of the above is true for all values of θ_0 with $0 < \theta_0 < \pi/2$.

20. What is the maximum value of the tension in the rod?

    (A) mg
    (B) $2mg$
    (C) $mL\theta_0/{T_0}^2$
    (D) $mg \sin\theta_0$
    (E) $mg(3 - 2\cos\theta_0)$

21. The experiment is repeated with a new pendulum with a rod of length $4L$, using the same angle θ_0, and the period of motion is found to be T. Which of the following statements is correct?

    (A) $T = 2T_0$ regardless of the value of θ_0.
    (B) $T > 2T_0$ with $T \approx 2T_0$ if $\theta_0 \ll 1$.
    (C) $T < 2T_0$ with $T \approx 2T_0$ if $\theta_0 \ll 1$.
    (D) $T > 2T_0$ for some values of θ_0 and $T < 2T_0$ for other values of θ_0.
    (E) T_0 and T are undefined because the motion is not periodic unless $\theta_0 \ll 1$.

Solution

The mass accelerates towards the pivot due to centripetal acceleration; if its speed is v, this acceleration is

$$a_c = \frac{v^2}{L}$$

Two forces act in the radial direction; the tension in the rod, inward, and the radial component of gravity, outward. At an angle θ to the vertical the radial component of gravity is

$$F_{g,rad} = mg\cos\theta$$

From Newton's second law we thus have for the tension F

$$F - F_{g,rad} = ma_c$$

$$F = mg\cos\theta + m\frac{v^2}{L}$$

Both terms increase as the mass approaches the bottom of the swing, so the maximum tension certainly occurs there. At that point the mass has traveled a vertical distance $L(1-\cos\theta_0)$, so from conservation of energy

$$\frac{1}{2}mv^2 = mgL(1-\cos\theta_0)$$

and so at the bottom

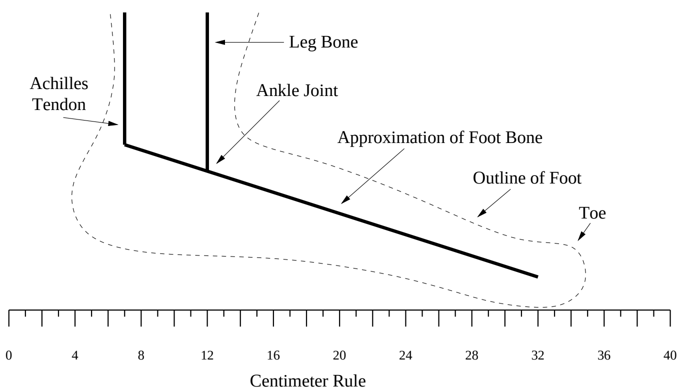
$$F = mg + 2mg(1-\cos\theta_0)$$
$$F = mg(3-2\cos\theta_0)$$

The motion is certainly periodic (one argument considers the fact that energy is conserved and the particle eventually returns to rest). From dimensional analysis the period must take the form

$$T = f(\theta_0)\sqrt{\frac{l_0}{g}}$$

Therefore, for any fixed θ_0, the period goes as $\sqrt{l_0}$.

22. A simplified model on the foot is shown. When a student of mass $m = 60$ kg stands on a single toe, the tension T in the Achilles Tendon is closest to



(A) $T = 600$ N

(B) $T = 1200$ N

(C) $T = 1800$ N

(D) $T = 2400$ N

(E) $T = 3000$ N

Solution

The entire weight of the student, $W = mg = 600$ N, is supported by the toe. Balancing torques about the ankle,

$$T \cdot 5 \text{ cm} = 600 \text{ N} \cdot 20 \text{ cm}$$
$$T = 2400 \text{ N}$$

The following information applies to questions 23 and 24

A man with mass m jumps off of a high bridge with a bungee cord attached to his ankles. The man falls through a maximum distance H at which point the bungee cord brings him to a momentary rest before he bounces back up. The bungee cord is perfectly elastic, obeying Hooke's force law with a spring constant k, and stretches from an original length of L_0 to a final length $L = L_0 + h$. The maximum tension in the Bungee cord is four times the weight of the man.

23. Determine the spring constant k.

    (A) $k = \frac{mg}{h}$
    (B) $k = \frac{2mg}{h}$
    (C) $k = \frac{mg}{H}$
    (D) $k = \frac{4mg}{H}$
    (E) $k = \frac{8mg}{H}$

24. Find the maximum extension of the bungee cord h.

    (A) $h = \frac{1}{2}H$
    (B) $h = \frac{1}{4}H$
    (C) $h = \frac{1}{5}H$
    (D) $h = \frac{2}{5}H$
    (E) $h = \frac{1}{8}H$

Solution

At the moment of maximum extension, all of the gravitational potential energy has been converted to spring potential energy.

$$mgH = \frac{1}{2}kh^2$$

We are given that the maximum tension in the cord is four times the weight of the man; this occurs at the moment of maximum extension.

$$kh = 4mg$$

These can be solved to yield

$$k = \frac{8mg}{H}$$

$$h = \frac{1}{2}H$$

as well as

$$k = \frac{4mg}{h}$$

although the latter is not an available answer choice.

25. A box with weight W will slide down a 30° incline at constant speed under the influence of gravity and friction alone. If instead a horizontal force P is applied to the box, the box can be made to move *up* the ramp at constant speed. What is the magnitude of P?

(A) $P = W/2$
(B) $P = 2W/\sqrt{3}$
(C) $P = W$
(D) $P = \sqrt{3}W$
(E) $P = 2W$

Solution

If block slides down at constant speed, then

$$\tan\theta = \mu$$

If block is pushed by horizontal force P up ramp at constant speed, then

$$F_N = mg\cos\theta + P\sin\theta$$

so friction is

$$F_f = \mu(mg\cos\theta + P\sin\theta)$$

This is balanced by gravity *and* the push up the ramp, so

$$P\cos\theta = \mu(mg\cos\theta + P\sin\theta) + mg\sin\theta$$

Divide through by $\cos\theta$, and use first equation

$$P = \mu(W + P\mu) + W\mu$$

or

$$P = \frac{2\mu}{1-\mu^2}W$$

or

$$P = \frac{2\frac{1}{\sqrt{3}}}{1-\frac{1}{3}}W$$