AAPT UNITED STATES PHYSICS TEAM
AIP 2019

2019 $F = ma$ Exam

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet. You may only use the scratch paper provided by the proctor.
- This test contains 25 multiple choice questions. Select the answer that provides the best response to each question. Please be sure to use a No.2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily of the same level of difficulty.
- Correct answers will be awarded one point; incorrect answers or leaving an answer blank will be awarded zero points. There is no additional penalty for incorrect answers.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- The question booklet, the answer sheet and the scratch paper will be collected at the end of this exam.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 1, 2019.**

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We acknowledge the following people for their contributions to this year's exams (in alphabetical order):

JiaJia Dong, Mark Eichenlaub, Matthew Huang, David Jones, Abijith Krishnan, Daniel Longenecker, Marianna Mao, Joon Pahk, Kye Shi, Brian Skinner, Alex Small, Paul Stanley, Elena Yudovina, and Kevin Zhou.

1. A uniform block of mass 10 kg is released at rest from the top of an incline of length 10 m and inclination 30°. The coefficients of static and kinetic friction between the incline and the block are $\mu_s = 0.15$ and $\mu_k = 0.1$. The end of the incline is connected to a frictionless horizontal surface. After a long time, how much energy is dissipated due to friction?

   (A) 75 J
   (B) 87 J ← **CORRECT**
   (C) 130 J
   (D) 147 J
   (E) 500 J

Solution

Because $mg\sin\theta$ is greater than the maximal static friction, the block will slide down the incline after release. The friction is kinetic friction. The block started with gravitational potential energy, $mgh = 10 \cdot 10 \cdot 5$, which is enough for the block to slide all the way down the incline. Thus the distance over which friction does work is the length of the incline.

$W_{\text{friction}} = \mu_k mg\cos\theta L = 0.1 \times 100 \times \frac{\sqrt{3}}{2} \times 10 = 87$ J.

2. Three cubical blocks of the same volume are made out of wood, styrofoam, and plastic. When the plastic block is placed in water, half of its volume is submerged. If the wooden block is placed in water with the plastic block on top, the wooden block is just fully submerged. Similarly, if the styrofoam block is placed in oil with the plastic block on top, the styrofoam block is just fully submerged. The density of oil is 0.7 times that of the water. What is the ratio of the density of wood to the density of styrofoam, $\rho_{\text{wood}}/\rho_{\text{styrofoam}}$?

   (A) 2.5 ← **CORRECT**
   (B) 0.5
   (C) 0.7
   (D) 1.4
   (E) 1

Solution

The buoyant force is $F_b = \rho_{\text{fluid}} g V_{\text{excluded}}$ and the gravity of an object is $mg = \rho V g$. For the plastic block,

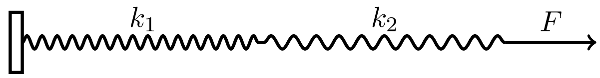
$$\rho_{\text{water}} g \frac{1}{2} V = \rho_{\text{plastic}} V g \to \rho_{\text{plastic}} = \frac{1}{2}\rho_{\text{water}}$$

Similarly, for the wood+plastic case:

$$\rho_{\text{plastic}} V + \rho_{\text{wood}} V = \rho_{\text{water}} V \to \rho_{\text{wood}} = 0.5\rho_{\text{water}}$$

and the styrofoam+plastic case yields $\rho_{\text{styrofoam}} = 0.2\rho_{\text{water}}$. Thus $\rho_{\text{wood}}/\rho_{\text{styrofoam}} = 2.5$.

3. Two springs of spring constants k_1 and k_2, respectively, are connected in series and stretched, as shown below. What is the ratio of their potential energies, U_1/U_2?



(A) 1
(B) k_1/k_2
(C) k_2/k_1 ← **CORRECT**
(D) $(k_1/k_2)^2$
(E) $(k_2/k_1)^2$

Solution

The tensions in the springs are equal, so $k_1 x_1 = k_2 x_2$. The potential energy in a spring is $\frac{1}{2}kx^2$, so the ratio is $\frac{U_1}{U_2} = \frac{k_1 x_1^2}{k_2 x_2^2}$. We can rewrite this as $\frac{U_1}{U_2} = \frac{k_2}{k_1}\left(\frac{k_1 x_1}{k_2 x_2}\right)^2$, which equals $\dfrac{k_2}{k_1}$.

Here is another way to think about the problem: the energy put into a spring is how much it stretches times the force to stretch it. The force is the same for the two springs in series, so the ratio of potential energies is the same as the ratio of how much they stretch, and that is inversely proportional to the spring constant by definition.

4. Consider two identical masses that interact only by gravitational attraction to each other. If one mass is fixed in place and the other is released from rest, then the two masses collide in time T. If both masses are released from rest, they collide in time

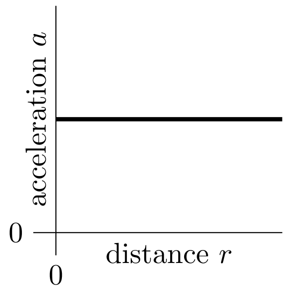
(A) $T/4$
(B) $T/(2\sqrt{2})$
(C) $T/2$
(D) $T/\sqrt{2}$ ← **CORRECT**
(E) T

Solution

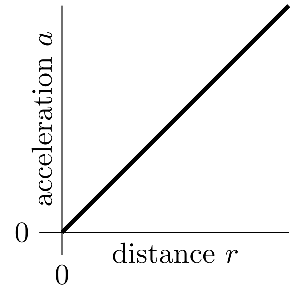
Suppose that when the masses are separated by a distance r, and one mass is held fixed, the other mass acquires a speed v from gravity. When both masses can move, they share the kinetic energy, so both have speed $v/\sqrt{2}$, so the relative speed is $\sqrt{2}v$. Hence to collapse the same distance r, the latter case will be $\sqrt{2}$ times faster, thus the time will be $T/\sqrt{2}$.

5. The density of the Earth increases gradually from around 3 g/cm^3 at the crust to about 13 g/cm^3 at the core. Which one of these plots could show local gravitational acceleration as a function of distance from Earth's center?

(A)



(B)



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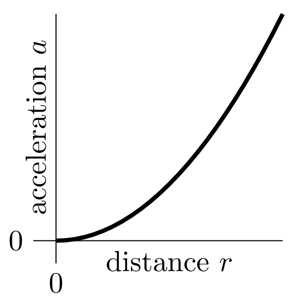
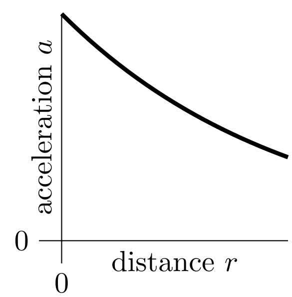
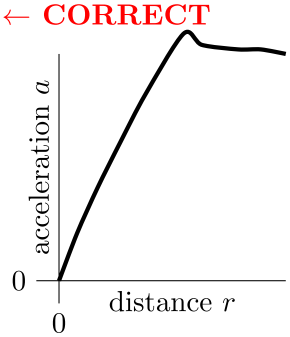


(D)



(E)

← **CORRECT**



Solution

The gravitational acceleration is zero at the center of the Earth by symmetry, so choices (A) and (D) are wrong. If the density is constant, the mass enclosed within a sphere of radius r grows as r^3. We also know $a \sim Mr^{-2}$, so $a \sim r$. This means choice (B) is for a constant-density planet. In Earth, density decreases with r, we know M grows more slowly than r^3, meaning the curve of a vs r must grow more slowly than linearly (and may decrease), so only (E) is possible.

6. A very long cylinder of dust is spinning about its axis with angular velocity ω at steady state. Let r be the distance from the axis. If the dust is only held together by gravity, the density of the dust is proportional to:

   (A) r^{-2}

   (B) r^{-1}

   (C) the density does not depend on r. $\leftarrow$ **CORRECT**

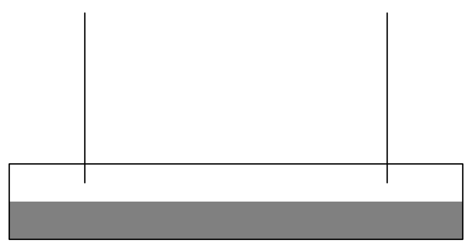
   (D) r

   (E) r^2

Solution

By Gauss's law for gravity, the gravitational acceleration at a cylindrical Gaussian surface is proportional to λ/r, where λ is the mass per length enclosed. The required centripetal acceleration is $a = \omega^2 r \propto r$. Hence we must have $\lambda \propto r^2$, which is achieved if ρ does not depend on r, as in this case $\lambda = \pi r^2 \rho$.

7. A trough half-filled with water is suspended from wires, as shown. The tension is initially the same in each wire.



A boat is placed in the trough directly under the left wire. It floats without touching the sides of the trough or overflowing the water. How does the tension in the wires change as a result?

   (A) The tension is not affected in either wire.

   (B) The tension increases equally in both wires. $\leftarrow$ **CORRECT**

   (C) The tension increases in the left wire and decreases in the right wire.

   (D) The tension increases in the left wire and stays the same in the right wire.

   (E) The tension increases in the left wire and increases by a smaller amount in the right wire.

Solution

The boat doesn't touch the wires or the trough. It only touches the water. The pressure at the bottom of the water remains the same everywhere, whether the boat is over top or not, because if the pressure were unequal, it would cause water to flow, but the situation is static. Therefore, the force from the water is distributed evenly over the trough and the tension is the same in each wire. The wires do support the boat-trough-water system, which got heavier, so the tension in the wires increases. The answer is B.

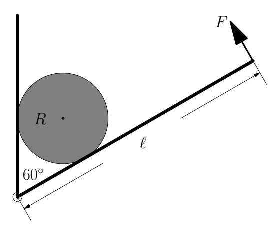
8. A scale is calibrated so that it gives a correct reading when sitting on the ground. A person holds the scale and presses it on both sides with their hands, pushing up on the bottom with the left hand and pushing down on the top with the right. The scale has a mass of 5 kg, and their left hand exerts a force of 200 N. The reading on the scale is:

   (A) 15 kg ← **CORRECT**
   (B) 20 kg
   (C) 35 kg
   (D) 40 kg
   (E) 45 kg

Solution

If the scale is correctly calibrated, the reading indicates the normal force acting on the top of the scale. By force balance, this is $200 - 50 = 150$ N, which is displayed as 15 kg.

9. A light board of length ℓ is hinged to a vertical wall. A solid ball of weight G and radius R is held between the wall and the board by a force F applied perpendicular at the end of the board, as shown in the figure below. Both the wall and the board are frictionless. The angle between the board and the wall is 60°. What is the magnitude of F?

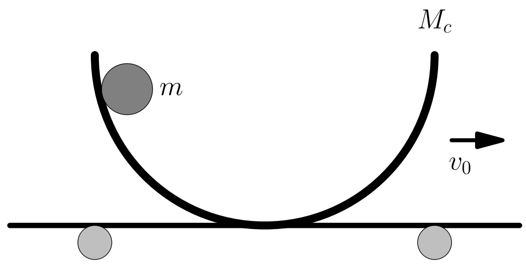


   (A) $\frac{2}{\sqrt{3}}G$
   (B) $\frac{2R}{\ell}G$ ← **CORRECT**
   (C) $\frac{2R}{\sqrt{3}\ell}G$
   (D) $\frac{4R}{\sqrt{3}\ell}G$
   (E) $\frac{\sqrt{3}R}{\ell}G$

Solution

We balance torques on the board about the hinge. The normal force of the board on the ball must have an upward component of G, and hence a magnitude of $2G/\sqrt{3}$ by basic trigonometry. Similarly, the lever arm of this torque is $\sqrt{3}R$. Then $F\ell = 2RG$, which gives $F = 2RG/\ell$.

10. A ball of mass m rolls in a bowl bolted on a cart. The bowl and cart move together and have a combined mass M_c, as shown in the figure. Initially, the cart moves to the right with speed v_0 with respect to the ground, while the ball is at rest with respect to the cart. The ball then slides down to the bottom of the bowl. When it reaches the bottom of the bowl, it moves with velocity v_b with respect to the cart. At this instant, how fast is the cart moving with respect to the ground?



(A) $v_0 - v_b$

(B) $v_0 - \dfrac{m}{M_c} v_b$

(C) $\dfrac{M_c v_0 - m v_b}{M_c + m}$

(D) $\dfrac{(m + M_c) v_0 - m v_b}{m + M_c}$ $\leftarrow$ **CORRECT**

(E) The answer depends on the coefficient of friction between the ball and bowl.

Solution

The problem can be solved using only momentum conservation, without knowing the friction between the ball and bowl. Work in the frame originally moving with the cart, where the momentum is zero. When the ball is at the bottom of the bowl, the ball and bowl have relative velocity v_b. This is only consistent with momentum conservation if the velocity of the bowl and ball are

$$-\frac{m}{m + M_c} v_b, \quad \frac{M_c}{m + M_c} v_b$$

respectively. Transforming back to the original frame, the final velocity of the bowl is

$$v_0 - \frac{m}{m + M_c} v_b.$$

11. Consider a flat uniform square of mass M and side length L. Cut a circle out of the square that has a diameter equal to the length of the side of the square, with the same center as the square. Determine the moment of inertia of the remaining shape about an axis through the center and perpendicular to the plane of the square.

    (A) $\left(\frac{1}{6} - \frac{\pi}{32}\right) ML^2$ ← **CORRECT**

    (B) $\left(\frac{1}{12} - \frac{\pi}{64}\right) ML^2$

    (C) $\left(\frac{\pi}{24} - \frac{1}{3\pi}\right) ML^2$

    (D) $\left(\frac{1}{2\pi} - \frac{1}{16}\right) ML^2$

    (E) $\left(\frac{1}{2\pi} - \frac{1}{8}\right) ML^2$

Solution

The moment of inertia of the square about a line that divides it into two equal rectangles is

$$I_1 = \frac{1}{12} ML^2$$

By perpendicular axis theorem, moment of inertia of a square about an axis through center of mass perpendicular to the plane is twice this, or

$$I_2 = \frac{1}{6} ML^2$$

Moment of inertia of the removed circle is

$$I_3 = \frac{1}{2} M_c (L/2)^2 = \frac{1}{8} M_c L^2,$$

where

$$M_c = \frac{\pi}{4} M$$

so

$$I = \frac{1}{2}\left(\frac{1}{3} - \frac{\pi}{16}\right) ML^2$$

12. Two planets A and B have masses $m_A = 2m_B$. They orbit a star in circular orbits of radius $r_A = 3r_B$. Let E_i and L_i be the kinetic energy and the magnitude of the angular momentum of planet i, respectively. Which of the following is true?

    (A) $E_A > E_B$ and $L_A > L_B$

    (B) $E_A > E_B$ and $L_A < L_B$

    (C) $E_A < E_B$ and $L_A > L_B$ ← **CORRECT**

    (D) $E_A < E_B$ and $L_A < L_B$

    (E) $E_A = E_B$ and $L_A > L_B$

Solution

For a mass m in a circular orbit of radius r, the kinetic energy is given by

$$E = \frac{GMm}{2r} \propto \frac{m}{r},$$

and the angular momentum is given by

$$L = m\sqrt{GMr} \propto m\sqrt{r}.$$

Thus

$$E_A = \frac{2}{3}E_B < E_B,$$

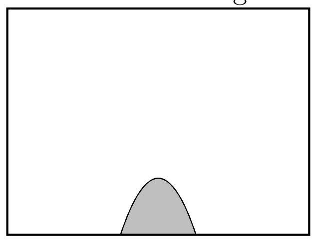
and

$$L_A = 2\sqrt{3}L_B > L_B.$$

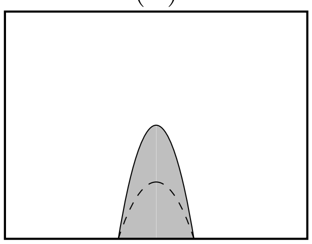
13. A water hose is at ground level against the base of a large wall. By aiming the hose at some angle, and squirting water at a speed v, one wets a region on the wall, as shown below. If the speed of the water is doubled, what is the new region that can be wetted? Ignore the effect of water splashing beyond the point of contact.

In the answer choices, the dotted line marks the initial wetted region.

initial wetted region

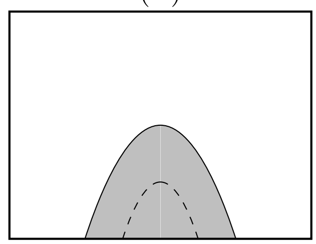


(A)



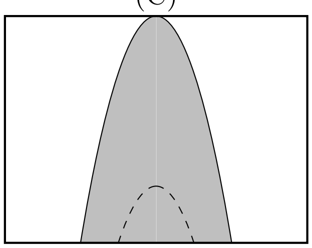
double height, same width

(B)



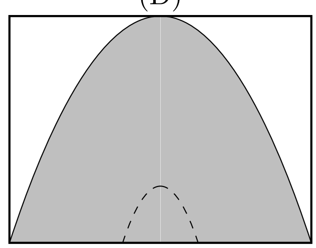
double height, double width

(C)



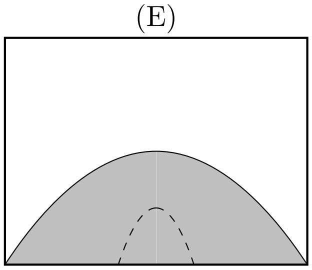
four times height, double width

(D)



four times height, four times width ← **CORRECT**

(E)



double height, four times width

Solution

The only dimensionful quantities in the problem are v and g, so all distances must be proportional to v^2/g. Hence the entire shape must scale as the velocity squared.

14. A ball is launched at speed v at angle θ above the horizontal toward a vertical wall a distance L away. It bounces elastically off the wall and falls back to its launch point. What was its initial speed?

(A) $\dfrac{\sqrt{Lg}}{\tan\theta}$

(B) $\dfrac{\sqrt{Lg}}{\sqrt{\tan\theta}}$

(C) $\dfrac{\sqrt{Lg}}{\sin\theta}$

(D) $\dfrac{\sqrt{Lg}}{\sqrt{\sin\theta}}$

(E) $\dfrac{\sqrt{2Lg}}{\sqrt{\sin 2\theta}}$ ← **CORRECT**

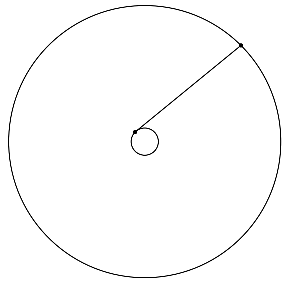
Solution

Since the collision reflects the horizontal velocity, this is equivalent to launching a ball with range $2L$. Using the range equation,

$$2L = \frac{v^2 \sin 2\theta}{g}$$

which gives $v = \sqrt{2Lg/\sin 2\theta}$.

15. A mass of $M = 100$ g is attached to the end of a string of length $R = 2$ m. A person swings the mass overhead such that their hand traverses a circle of radius $r = 3$ cm at angular velocity $\omega = 10$ rad/s, ahead of the mass M by an angle of $\pi/2$. Estimate the force of air resistance on the object.



(A) 0.02 N

(B) 0.03 N

(C) 0.2 N

(D) 0.3 N ← **CORRECT**

(E) 2 N

Solution

The tension in the string is approximately

$$T = M\omega^2 R = 20\,\text{N}.$$

Since the mass and string have constant kinetic energy, the power delivered by the hand must balance air resistance. Using $P = Fv$, we have

$$T\omega r = F\omega R$$

where F is the force of air resistance. Then $F = Tr/R = 0.3$ N.

The following information applies to 16 and 17.

A hoop of radius r is launched to the right at initial speed v_0 at ground-level. As it is launched, it is also spun counterclockwise at angular velocity $3v_0/r$. The coefficient of kinetic friction between the ground and the hoop is μ_k.

16. How long does it take the hoop to return to its starting position?

(A) $\dfrac{v_0}{2\mu_k g}$

(B) $\dfrac{2v_0}{\mu_k g}$ ← **CORRECT**

(C) $\dfrac{\mu_k v_0}{2g}$

(D) $\dfrac{\mu_k v_0}{g}$

(E) $\dfrac{2\mu_k v_0}{g}$

Solution

As long as the hoop is slipping against the ground, there is a constant leftward force of $F = \mu_k N = \mu mg$ acting on the hoop. Thus, the hoop undergoes uniformly-accelerated motion while it is slipping.

We can find the time T after which the hoop stops slipping:

$v(T) = r\omega(T) = v_0 - \mu g T = r(-3v_0/r + \mu g T/r)$

Rearranging, $T = \frac{2v_0}{\mu g}$.

Alternatively, we can observe that there are no net torque on the hoop about a fixed point on the ground, and therefore the angular momentum is conserved:

$$mv_0 r + (mr^2)(-3v_0/r) = (mr^2 + mr^2)(v_f/r),$$

which gives

$$v_f = -v_0.$$

Since the hoop was uniformly accelerated up to this point, the position of the hoop must be the same as when it started. So T is the desired time, and thus the solution is $T = \frac{2v_0}{\mu g}$.

17. Would the answer to the previous question, namely the amount of time to return to the starting position, be the same for a uniform disk with the same coefficient of friction launched with the same initial conditions?

(A) Yes.

(B) No, because the torque due to friction is larger for the hoop.

(C) No, because the disk never returns to its starting position.

(D) No, because the disk stops slipping too soon. ← **CORRECT**

(E) There is not enough information to decide.

Solution

In this case, $I = \frac{1/2}{m} r^2 < mr^2$, so the equation for the time before the disk stops slipping is:
$v(T) = r\omega(T) = v_0 - \mu g T = r(-3v_0/r + 2\mu g T/r)$,
So $T = \frac{4v_0}{3\mu g}$, and $v(T) = -\frac{v_0}{3}$. Thus, the disk stops slipping before returning to its starting position.

18. You are predicting the time for a car to complete a race. The car begins with an initial speed v_0 and maintains a constant acceleration a throughout the race. Both v_0 and a have independent uncertainties of 10%. Which contributes greater uncertainty to your estimate of the time for the car to complete the race?

(A) The uncertainty in v_0 for sufficiently short races and the uncertainty in a for sufficiently long races. ← **CORRECT**

(B) The uncertainty in v_0 for sufficiently long races and the uncertainty in a for sufficiently short races.

(C) The uncertainty in v_0.

(D) The uncertainty in a.

(E) They contribute equally in the uncertainty.

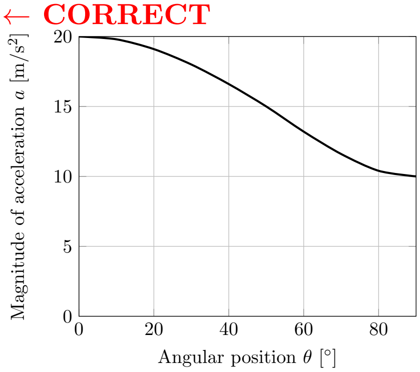
Solution

The average speed during the race is $v_0 + \dfrac{at}{2}$, where t is the time for completion of the race. The total uncertainty is $\sigma_t = \sqrt{\sigma_{v_0}^2 + \left(\sigma_a \dfrac{t}{2}\right)^2}$. The more important uncertainty is whichever is larger between σ_{v_0} and $\sigma_a \dfrac{t}{2}$. For sufficiently short races, $at \ll v_0$ and the car does not accelerate appreciably, so uncertainty in the acceleration is unimportant. For sufficiently long races, $at \gg v_0$ and uncertainty in the initial speed is unimportant, so the answer is D.

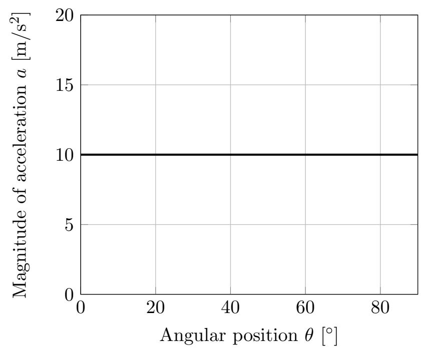
19. A physical pendulum consists of a mass on one end of a massless rigid rod that can pivot about the opposite end. Assuming the oscillations have an amplitude of 90°, which of the following graphs best shows the total acceleration of the mass as a function of the angular position θ, measured between the pendulum and the vertical direction?

(A)

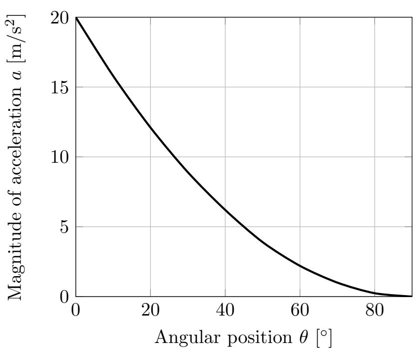
← **CORRECT**



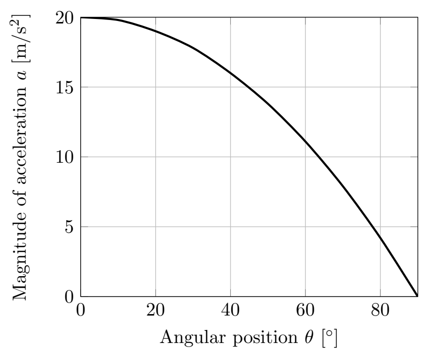
(B)



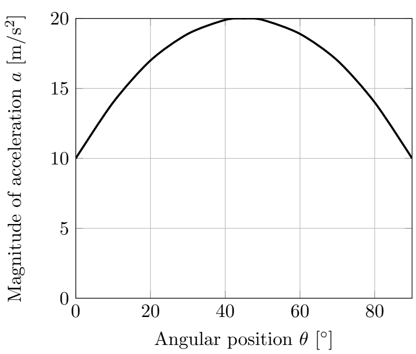
(C)



(D)



(E)



Solution

The tangential component of acceleration is of magnitude

$$a_\theta = g\sin\theta$$

The radial component is

$$a_r = v^2/r$$

But if it is released at 90°, then

$$\frac{1}{2}v^2 = gr\cos\theta$$

so

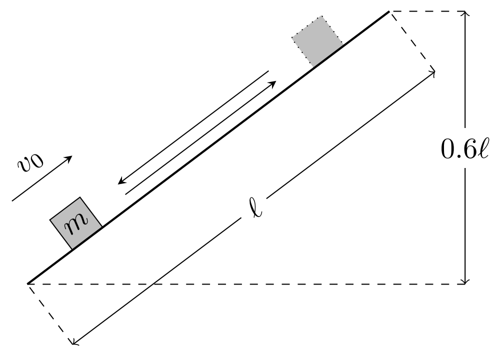
$$a_r = 2gr\cos\theta$$

$$a^2 = g^2\left(1 + 3\cos^2\theta\right)$$

It can also be solved partly by inspection: When $\theta = 90°$ the mass is at momentarily at rest, and is accelerating down at g. This eliminates (C) and (D). However, one must be careful about the remaining choices. At the bottom, the acceleration is just the centripetal acceleration. By energy conservation, the speed at the bottom is $v = \sqrt{2gR}$, where R is the length of the pendulum arm, so the acceleration is $a = a_c = v^2/R = 2g = 20\,\mathrm{m\,s^{-2}}$.

20. A block on a ramp is given an initial velocity v_0 upward along the ramp. It slides upward for a time t_u, traveling some distance, and then slides downward for a time t_d until it returns to its original position. What is t_d in terms of t_u?

The height of the incline is 0.6 times its length and the coefficient of kinetic friction between the block and the incline is 0.5.



(A) $t_0/5$

(B) $t_0/\sqrt{5}$

(C) $\sqrt{5}t_0$ ← **CORRECT**

(D) $2t_0$

(E) $5t_0$

Solution

As the block slides upward, friction and gravity both pull the block downward, the block has a constant deceleration (i.e. acceleration along the downhill direction)

$$a_u = g\sin\theta + \mu g\cos\theta,$$

where θ is the inclination angle and $\mu = 0.5$ is the coefficient of friction.

As the block slides downward, gravity pulls downward but friction resists the downward motion, so the block has an acceleration (along the downhill direction)

$$a_d = g\sin\theta - \mu g\cos\theta.$$

The time to travel a distance d under constant acceleration a, starting (or ending) at rest, is

$$t = \sqrt{\frac{2d}{a}},$$

so the ratio of travel times is

$$\frac{t_d}{t_u} = \sqrt{\frac{a_u}{a_d}} = \sqrt{\frac{\sin\theta + \mu\cos\theta}{\sin\theta - \mu\cos\theta}} = \sqrt{5}.$$

21. An empty freight car on a level railroad track has a mass M. A chute above the freight car opens and grain falls down into the car at a rate of r, measured in kilograms per second. The grain falls a vertical distance h before hitting the bed of the freight car without bouncing up. How much normal force is exerted on the freight car from the rails at a time t after the grain begins to hit the bed of the car? Assume the grain starts from rest.

    (A) $Mg + rtg$

    (B) $Mg + r\sqrt{gh}$

    (C) $Mg + r\sqrt{2gh}$

    (D) $Mg + r\sqrt{gh} + rtg$

    (E) $Mg + r\sqrt{2gh} + rtg$ ← **CORRECT**

Solution

The (upward) normal force on the freight car must equal the total downward forces experienced by the freight car. The freight car experiences three downward forces:

- The freight car's own weight:
$$F_{g,\text{car}} = Mg.$$
- The weight of the grain accumulating in the freight car. The grain accumulates at a (mass) rate r, so after a time t the car contains a total grain mass rt. Thus the total weight of the grain after time t is
$$F_{g,\text{grain}} = rtg.$$
- The force from absorbing the momentum of the falling grain when the grain lands on the cart. Over some time interval Δt, an amount of grain with mass $\Delta m = r\Delta t$ falls onto the cart. The grain falls from a height h, so it lands on the cart with speed $v = \sqrt{2gh}$. Thus the grain carries a momentum
$$\Delta p = \Delta m v = r\Delta t\sqrt{2gh}.$$
Thus the rate at which the momentum is delivered to the cart, or the force experienced by the cart, is
$$F_{\text{collision}} = \frac{\Delta p}{\Delta t} = \frac{r\Delta t\sqrt{2gh}}{\Delta t} = r\sqrt{2gh}.$$

Thus the normal force, which equals the total downward force on the cart, is

$$F_{g,\text{car}} + F_{g,\text{grain}} + F_{\text{collision}} = Mg + rtg + r\sqrt{2gh}.$$

22. The bruise threshold for a fruit is the largest height it can be dropped from rest without bruising. The bruise threshold of a 0.2 kg apple on steel is 10 cm. Suppose that it always takes 0.1 seconds after impact for the apple to fully stop and that 4 cm^2 of the apple comes into contact with the surface during the impact. What minimum average pressure is required to cause the apple to bruise?

(A) 67 Pa

(B) 210 Pa

(C) 7100 Pa ← **CORRECT**

(D) 23 000 Pa

(E) 67 000 Pa

Solution

An apple with mass $m = 0.2\,\mathrm{kg}$ bruises when dropped from a height h hits the ground with a speed $v = \sqrt{2gh}$ and thus carries momentum

$$p = mv = m\sqrt{2gh}.$$

If it takes $t = 0.1\,\mathrm{s}$ to fully stop, it experiences an average force of

$$F = \frac{p}{t} = \frac{m\sqrt{2gh}}{t}.$$

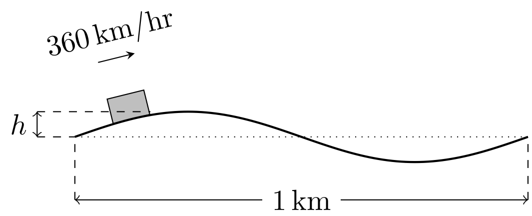
Over a contact area $A = 4\,\mathrm{cm}^2 = 4 \times 10^{-4}\,\mathrm{m}^2$, the average pressure would be

$$P = \frac{F}{A} = \frac{m\sqrt{2gh}}{At}.$$

The apple bruises when the height from which the apple is dropped exceeds the bruise threshold, i.e. $h > h_{\text{threshold}} = 10\,\mathrm{cm} = 0.10\,\mathrm{m}$. Correspondingly, the apple bruises when the average pressure exceeds

$$P > P_{\text{min}} = \frac{m\sqrt{2gh_{\text{threshold}}}}{At} = \frac{(0.2\,\mathrm{kg})\sqrt{2(10\,\mathrm{m\,s^{-2}})(0.10\,\mathrm{m})}}{(4 \times 10^{-4}\,\mathrm{m}^2)(0.1\,\mathrm{s})} = 7100\,\mathrm{Pa}.$$

23. A train travels at 360 km/hr on an almost straight track. The track is slightly sinusoidal, with a vertical amplitude of h over a 1 km distance, as shown in the figure. If the maximum tolerable vertical acceleration of the train is set at 0.1 m/s^2, what is the maximum allowable size of h?



(A) 12.5 cm

(B) 25 cm ← **CORRECT**

(C) 50 cm

(D) 100 cm

(E) 1000 cm

Solution

The train travels one full wavelength of track over a time

$$T = \frac{\lambda}{v},$$

where $\lambda = 1\,\mathrm{km}$ is the wavelength of the track and $v = 360\,\mathrm{km\,h^{-1}} = 0.1\,\mathrm{km\,s^{-1}}$ is the speed of the train. During this time, the train undergoes one full period of vertical oscillation. Thus the period of vertical oscillation is $T = \frac{\lambda}{v}$, with angular frequency

$$\omega = \frac{2\pi}{T} = \frac{2\pi v}{\lambda}.$$

The amplitude of vertical oscillation is h, so the maximum vertical acceleration is

$$a = \omega^2 h = \left(\frac{2\pi v}{\lambda}\right)^2 h.$$

The maximum vertical acceleration must be less than or equal to the maximum tolerable acceleration $a_{\text{tolerable}} = 0.1\,\mathrm{m\,s^{-2}}$:

$$a = \left(\frac{2\pi v}{\lambda}\right)^2 h \le a_{\text{tolerable}}.$$

Thus the maximum allowable vertical amplitude is

$$h \le \left(\frac{\lambda}{2\pi v}\right)^2 a_{\text{tolerable}} = \left(\frac{(1\,\mathrm{km})}{2\pi(0.1\,\mathrm{km\,s^{-1}})}\right)^2 (0.1\,\mathrm{m\,s^{-2}}) = 0.25\,\mathrm{m} = 25\,\mathrm{cm}.$$

24. A uniform rope of length L and mass M passes over a frictionless pulley, and hangs with both ends at equal heights. If one end is pulled down a distance x and the rope is released, the acceleration of that end at this instant will be:

   (A) $xg/4L$
   (B) $xg/3L$
   (C) $xg/2L$
   (D) xg/L
   (E) $2xg/L$ ← **CORRECT**

Solution

When the ends of the rope hang on the pulley at equal heights, the length of the rope is $L/2$ on both sides.

If one end of the rope is pulled down a distance x, the length of rope on that side is $L/2 + x$, and the remaining length of rope on the other side is $L/2 - x$. Thus the difference in lengths is

$$\Delta l = (L/2 + x) - (L/2 - x) = 2x,$$

and thus the mass of rope on one side of the pulley exceeds the mass of rope on the other side by an amount

$$\Delta m = M\left(\frac{l}{L}\right) = 2M\left(\frac{x}{L}\right),$$

and so the rope experiences a net force due to gravity along the "rope direction"

$$F = \Delta m g = 2Mg\left(\frac{x}{L}\right)$$

and hence acceleration

$$a = \frac{F}{M} = 2g\left(\frac{x}{L}\right) = 2xg/L.$$

25. A student makes an estimate of the acceleration due to gravity, g, by dropping a rock from a known height h and measuring the time, t, it takes to hit the ground. Neglecting air resistance, which one of the following situations will lead to the smallest value of the relative uncertainty, $(\Delta g)/g$, in the estimate?

   (A) There is no uncertainty in t, and h has a 10% uncertainty. ← **CORRECT**
   (B) There is no uncertainty in h, and t has a 10% uncertainty.
   (C) Both t and h have a 5% uncertainty.
   (D) (A) and (B) yield the same uncertainty, which is smaller than in (C).
   (E) (A), (B), and (C) all yield the same uncertainty.

Solution

When a rock is released from a height h, the time t it takes to hit the ground is given by $h = \frac{1}{2}gt^2$.

This means that the acceleration due to gravity is estimated using the equation

$$g = \frac{2h}{t^2}. \tag{1}$$

Since t is squared in this expression while h appears only in the first power, uncertainty in t has a larger effect on the estimate of g than uncertainty in h. More quantitatively, the rules for error propagation are given in the solutions for $F = ma$ 2018 A. We will derive the relevant rules explicitly below:

Consider (A), the uncertainty Δg is given by

$$\Delta g = \left| \frac{dg}{dh} \right| \times \Delta h, \tag{2}$$

where $dg/dh = 2/t^2$ is the derivative of g with respect to h. Thus

$$\Delta g = \frac{2}{t^2} \Delta h. \tag{3}$$

Dividing both sides by $g = 2h/t^2$ gives

$$\frac{\Delta g}{g} = \frac{\Delta h}{h}. \tag{4}$$

So a 10% uncertainty in h implies a 10% uncertainty in g.

Now look at case (B),

$$\Delta g = \left| \frac{dg}{dt} \right| \times \Delta t, \tag{5}$$

where $dg/dt = 4h/t^3$ is the derivative of g with respect to t. Dividing both sides by $g = 2h/t^2$ gives

$$\frac{\Delta g}{g} = 2\frac{\Delta t}{t}. \tag{6}$$

So a 10% uncertainty in t implies a 20% uncertainty in g.

If both Δt and Δh are nonzero, then these two independent contributions to the uncertainty add together like the sides of a right triangle:

$$\left(\frac{\Delta g}{g}\right)^2 = \left(\frac{\Delta h}{h}\right)^2 + \left(2\frac{\Delta t}{t}\right)^2. \tag{7}$$

Inserting $(\Delta h)/h = (\Delta t)/t = 0.05$ gives $(\Delta g)/g = 0.112$, which means an 11.2% uncertainty. So we can conclude that situation (A) produces the smallest uncertainty in $(\Delta g)/g$.

Alternatively, you can arrive at the same conclusion by exploiting:

$$g = \frac{2h}{t^2},$$
$$\ln g = \ln 2 + \ln h - 2 \ln t.$$
$$\frac{\mathrm{d}g}{g} = \frac{\mathrm{d}h}{h} - 2\frac{\mathrm{d}t}{t}.$$