AAPT UNITED STATES PHYSICS TEAM
AIP 2026

USA Physics Olympiad Exam

Instructions for the Student

REFERENCE SHEET

Fundamental Constants

$g = 9.8$ N/kg

$G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2

$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9$ N $\cdot$ m^2/C^2

$k_{\mathrm{m}} = \mu_0/4\pi = 10^{-7}$ T $\cdot$ m/A

$c = 3.00 \times 10^8$ m/s

$k_{\mathrm{B}} = 1.38 \times 10^{-23}$ J/K

$N_{\mathrm{A}} = 6.02 \times 10^{23}$ (mol)$^{-1}$

$R = N_{\mathrm{A}} k_{\mathrm{B}} = 8.31$ J/(mol $\cdot$ K)

$\sigma = 5.67 \times 10^{-8}$ J/(s $\cdot$ m^2 $\cdot$ K^4)

$e = 1.602 \times 10^{-19}$ C

1 eV $= 1.602 \times 10^{-19}$ J

$h = 6.63 \times 10^{-34}$ J $\cdot$ s $= 4.14 \times 10^{-15}$ eV $\cdot$ s

$m_e = 9.109 \times 10^{-31}$ kg $= 0.511$ MeV/c^2

Useful Approximations

$$(1+x)^n \approx 1 + nx + n(n-1)x^2/2 \text{ for } |nx| \ll 1$$
$$e^x \approx 1 + x + x^2/2 + x^3/6 \text{ for } |x| \ll 1$$
$$\sin\theta \approx \theta - \theta^3/6 \text{ for } |\theta| \ll 1$$
$$\cos\theta \approx 1 - \theta^2/2 \text{ for } |\theta| \ll 1$$

You may use this sheet for both parts of the exam.

We acknowledge the US Physics Team coaches and other people for their contributions to this year's exam (in alphabetical order):

Tengiz Bibilashvili, Eddie Chen, Kellan Colburn, Evan Erickson, Harry Kroft, Natalie LeBaron, Rishab Parthasarathy.

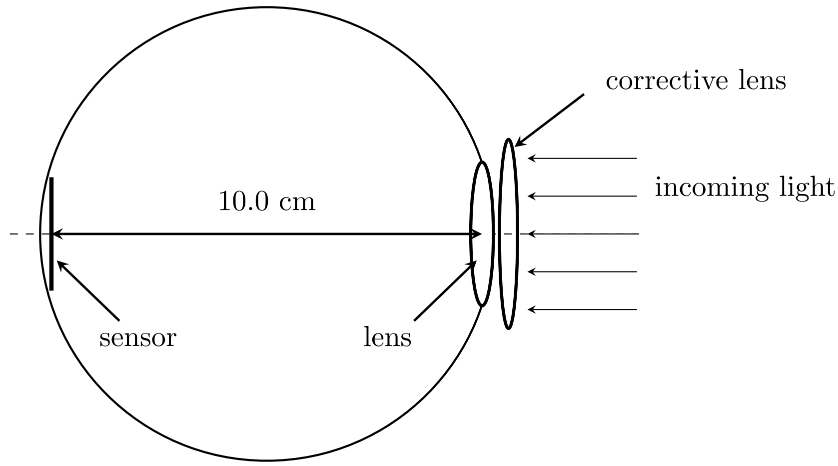
Problem A1: 404 Vision Not Found

A robot is equipped with an eye containing an adjustable thin lens mounted at the front. A small flat retina-like sensor is mounted on the optical axis at the back of the eye. The distance from the lens to the sensor is

$$s' = 10.0 \text{ cm}.$$

The circular aperture of the lens (the pupil) has diameter

$$d = 5.00 \text{ mm}.$$



By adjusting its internal lens, the robot can focus sharply on objects whose distances lie between

$$1.00 \text{ m} \leq s \leq 10.0 \text{ m}.$$

In this problem, treat all lenses as thin.

(a) Find both the range of focal lengths f and the range of optical powers P through which the internal lens is adjustable.

Solution

The image distance is fixed at

$$s' = 0.10 \text{ m}.$$

The thin lens equation gives

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}.$$

Since optical power is

$$P = \frac{1}{f},$$

we have

$$P = \frac{1}{s} + \frac{1}{s'}.$$

For

$$s = 10.0 \text{ m},$$

$$P = \frac{1}{10.0} + \frac{1}{0.10} = 0.10 + 10.0 = 10.1 \text{ D}.$$

For

$$s = 1.00 \text{ m},$$

$$P = \frac{1}{1.00} + \frac{1}{0.10} = 1.0 + 10.0 = 11.0 \text{ D}.$$

Thus,

$$P_{\text{eye}} \in [10.1,\ 11.0] \text{ D}.$$

Since

$$f = \frac{1}{P},$$

the focal-length range is

$$f \in [9.09,\ 9.90] \text{ cm}.$$

The robot is to be adapted for two new functions:

- observing stars (objects at infinity),
- reading printed material at a distance of 25.0 cm.

To do this, a thin corrective lens is placed immediately in front of the eye. Neglect the separation between this corrective lens and the eye lens.

(b) Find the range of optical powers P of the corrective lens that allows the robot to observe stars.

Solution

For stars,

$$s = \infty,$$

so the corrective lens must form a virtual image at a distance L that lies within the robot's normal focusing range:

$$1.00 \text{ m} \leq L \leq 10.0 \text{ m}.$$

For the corrective lens,

$$P = \frac{1}{s} - \frac{1}{L}.$$

Since $1/s = 0$,

$$P = -\frac{1}{L}.$$

Thus,

$$P_{\text{corr}} \in [-1.0,\ -0.10] \text{ D}.$$

(c) Can a single corrective lens allow the robot to both observe stars and read at 25.0 cm? Explain your answer.

Solution

To read at 25.0 cm, the total optical power required is

$$P_{\text{total}} = \frac{1}{0.25} + \frac{1}{0.10} = 4 + 10 = 14 \text{ D}.$$

The internal lens can provide powers in the range

$$P_{\text{eye}} \in [10.1,\ 11.0] \text{ D}.$$

Thus, the corrective lens must provide

$$P_{\text{corr}} = P_{\text{total}} - P_{\text{eye}},$$

which gives

$$P_{\text{corr}} \in [14 - 11.0,\ 14 - 10.1] = [3.0,\ 3.9] \text{ D}.$$

From part (b), observing stars requires

$$P_{\text{corr}} \in [-1.0,\ -0.10] \text{ D}.$$

These intervals do not overlap, so no single corrective lens can perform both functions.

(d) Estimate the diffraction-limited maximum print resolution (in dots per inch, dpi) when viewing a page at 25.0 cm. Use light of wavelength

$$\lambda = 550 \text{ nm}.$$

Solution

The diffraction-limited angular resolution is approximately

$$\theta \approx 1.22\,\frac{\lambda}{d}.$$

Using

$$\lambda = 5.5 \times 10^{-7} \text{ m}, \qquad d = 5.0 \times 10^{-3} \text{ m},$$

we get

$$\theta \approx 1.22 \cdot \frac{5.5 \times 10^{-7}}{5.0 \times 10^{-3}} \approx 1.34 \times 10^{-4} \text{ rad}.$$

At a viewing distance

$$L = 0.25 \text{ m},$$

the smallest resolvable spacing is

$$\delta x = L\theta = 0.25 \cdot 1.34 \times 10^{-4} \approx 3.35 \times 10^{-5} \text{ m}.$$

Thus,

$$\text{dpi} = \frac{0.0254}{\delta x} \approx 760.$$

(e) When viewing a stellar object, the eye lens is adjusted so that the image forms on the sensor. The total optical power collected by the pupil is P_0. Due to diffraction at the pupil, the image is a central bright spot (Airy disk).

(i) Estimate the intensity at the sensor in terms of P_0, λ, and the parameters of the eye.

(ii) By what factor does the intensity change if, while observing the same object, the pupil diameter is reduced from 5.0 mm to 2.0 mm?

Solution

(i) The diffraction-limited spot has characteristic size

$$r \sim \frac{\lambda s'}{d}.$$

Thus the area scales as

$$A \sim \frac{\lambda^2 s'^2}{d^2}.$$

Hence

$$I \sim \frac{P_0}{A} \sim \frac{P_0 d^2}{\lambda^2 s'^2}.$$

(ii) Since $P_0 \propto d^2$, we have

$$I \propto d^4.$$

Thus,

$$\frac{I_{\text{new}}}{I_{\text{old}}} = \left(\frac{2}{5}\right)^4 = \frac{16}{625} \approx 0.026,$$

so the intensity decreases by a factor of about 39.

Problem A2: Straight on a Cone

In general relativity, particles in the absence of external forces move "straight" along curved spacetime. In this problem, we will analyze a classical mechanics analogue of this idea.

Suppose a point mass m is constrained to move along the surface of an infinite, fixed, frictionless cone of half-angle α. Ignore gravity throughout this problem, and assume the only force on the mass is the normal force from the cone. Let the vertex of the cone be O.

At time $t = 0$, the mass is at a point A, a distance r_0 from O, and its velocity has magnitude v and is perpendicular to the displacement vector $\overrightarrow{OA}$ (i.e., tangent to the cone).

(a) Find the normal force at time $t = 0$.

Solution

At $t = 0$, the motion is along a circle of radius

$$\rho_0 = r_0 \sin\alpha$$

about the axis of the cone. Thus the required centripetal acceleration is

$$a_c = \frac{v^2}{\rho_0} = \frac{v^2}{r_0 \sin\alpha}.$$

The normal force N is the only force acting, and its horizontal component (after removing a component parallel to the cone) provides this centripetal acceleration. Since the normal makes an angle α with the horizontal plane,

$$\frac{N}{\cos\alpha} = m\frac{v^2}{r_0 \sin\alpha}.$$

Thus,

$$N = \frac{mv^2}{r_0} \cot\alpha.$$

(b) At some later time, the mass is at a point B, a distance r_1 from O. Find the angle ψ its velocity vector makes with the displacement vector $\overrightarrow{OB}$.

Solution

The component of angular momentum about the axis of the cone is conserved, since the normal force exerts no torque about that axis.

Let $\rho = r \sin\alpha$ be the distance from the axis. Then

$$L_z = m\rho v_\phi.$$

Initially,

$$L_z = m(r_0 \sin\alpha)v.$$

At point B, if the velocity makes an angle ψ with $\overrightarrow{OB}$, then the azimuthal component of velocity is

$$v_\phi = v \sin\psi.$$

Thus,

$$m(r_0 \sin\alpha)v = m(r_1 \sin\alpha)(v \sin\psi).$$

Cancelling common factors gives

$$\sin\psi = \frac{r_0}{r_1}.$$

Therefore,

$$\psi = \arcsin\left(\frac{r_0}{r_1}\right).$$

(c) At what time t does the mass reach the point B?

Solution

The cone can be unrolled into a flat plane. Since the normal force is always perpendicular to the surface, the particle moves in a straight line with constant speed v on this plane.

In the unrolled plane, the motion forms a right triangle with sides r_0 and r_1, so the distance traveled is

$$\sqrt{r_1^2 - r_0^2}.$$

Thus,

$$t = \frac{\sqrt{r_1^2 - r_0^2}}{v}.$$

(d) After a long time, find the total number of revolutions the mass makes around the axis of the cone.

Solution

On the unrolled plane, the trajectory is a straight line tangent to the circle of radius r_0. As $t \to \infty$, the angle swept in the plane is

$$\Delta\beta = \frac{\pi}{2}.$$

The relation between the angle β in the plane and the azimuthal angle ϕ on the cone is

$$\beta = \phi \sin\alpha.$$

Thus,

$$\Delta\phi = \frac{\Delta\beta}{\sin\alpha} = \frac{\pi}{2\sin\alpha}.$$

The total number of revolutions is

$$N = \frac{\Delta\phi}{2\pi} = \frac{1}{4\sin\alpha}.$$

Problem A3: Storm Chaser

The violent updrafts and vortices that power severe thunderstorms arise from atmospheric instability. Neglect moisture effects, and model the atmosphere as an ideal gas of molar mass μ. Let C_P and C_V be the Molar heat capacities at constant pressure and volume, respectively, with $C_P = C_V + R$. Gravitational acceleration is g.

Part A: Updraft

(a) Derive the relationship between atmospheric pressure and altitude, $\frac{dP}{dz}$, in terms of the atmospheric density ρ.

Solution

Force balance on an air slab of area A at z with thickness dz:

$$P(z)A = P(z + dz)A + \rho g A dz \quad (0\text{-}1)$$

$$\frac{dP}{dz} = -\rho g \quad (0\text{-}2)$$

(b) A parcel of air is lofted rapidly upward in a storm's updraft, exchanging no heat with its surroundings. Assume the parcel remains in pressure equilibrium with the surrounding atmosphere as it rises. Using your result from part (a) and the first law of thermodynamics, derive the rate at which the parcel's temperature changes with altitude, $\Gamma_p \equiv -\frac{dT_p}{dz}$.

Solution

For an adiabatic process $dU = -PdV$. With $\frac{dU}{dT} = nc_v$ we have:

$$nc_v dT = -PdV$$

Differentiating ideal gas law and substituting gives:

$$nc_v dT = VdP - nRdT \quad (0\text{-}3)$$

$$n(c_v + R)dT = nc_p dT = VdP \quad (0\text{-}4)$$

$$(0\text{-}5)$$

Substituting result from a and dividing by mass:

$$\frac{n}{m} c_p dT = -\frac{\rho g V}{m} dz \quad (0\text{-}6)$$

$$\frac{c_p}{\mu} dT = -g dz \quad (0\text{-}7)$$

$$\frac{dT}{dz} = -\frac{\mu g}{c_p} \equiv -\Gamma_p \quad (0\text{-}8)$$

(c) The atmosphere ahead of the storm has an environmental temperature profile $T_{env}(z)$ with lapse rate $\Gamma_{env} \equiv -\frac{dT_{env}}{dz}$ and surface temperature $T_{env}(0) = T_0$. A parcel in equilibrium at $z = 0$ is displaced upward by δz. Determine:

(i) The condition on Γ_{env} for the atmosphere to be unstable, and

(ii) The oscillation frequency of the parcel when the atmosphere is stable.

Solution

The temperature of the parcel after displacement is $T_p = T_0 - \Gamma_p \delta z$. The net force on the parcel per unit mass f is:

$$F = \rho_{env} V g - \rho_p V g \tag{0-9}$$

$$f = \frac{\rho_{env} - \rho_p}{\rho_p} g \tag{0-10}$$

For an ideal gas $\rho \sim \frac{P}{T}$. Since we assume the parcel remains in pressure equilibrium, $P_p = P_{env}$, so we can substitute ρ for $1/T$:

$$f = \frac{T_p - T_{env}}{T_{env}} g \approx \frac{T_p - T_{env}}{T_0} g = \frac{\left(\Gamma_{env} - \Gamma_p\right)}{T_0} g \delta z \tag{0-11}$$

So the atmosphere is unstable for $\Gamma_{env} > \Gamma_p$ and the oscillation frequency is $\sqrt{\frac{g}{T_0}\left(\Gamma_p - \Gamma_{env}\right)}$

Part B: Vortex

In this section, treat the atmosphere as isothermal with uniform density ρ_0 and ambient pressure P_∞.

(a) A tornado can be modeled as a Rankine vortex: solid body rotation $v = \Omega r$ for $r < R$ and irrotational flow $v = \Omega R^2/r$ for $r > R$, with $P \to P_\infty$ as $r \to \infty$. Using radial force balance on a small parcel of air moving in a circular path, derive the radial pressure gradient and find the central pressure deficit $\Delta P = P_\infty - P(0)$.

Solution

Analogously to part A(a), consider annulus of radius r cross section A and thickness dr:

$$P(r)A = P(r + dr)A + \rho_0 A \frac{v^2}{r} dr \tag{0-12}$$

$$\frac{dP}{dr} = \frac{\rho_0 v^2}{r} \tag{0-13}$$

Integrating the velocity profile in the outer region gives $P(r)_{r>R} = P_\infty - \frac{\rho_0 \Omega^2 R^2}{2}$. Integrating in the inner region and boundary matching gives:

$$\Delta P = P_\infty - P(0) = \rho_0 \Omega^2 R^2$$

(b) Using a displacement argument analogous to A(c), derive the condition on the circulation $\gamma(r) \equiv rv(r)$ for a rotating fluid to be stable. Apply your result to the Rankine vortex.

Solution

Displace the parcel $r \to r + \delta r$. Angular momentum conservation means $\gamma = rv$ is conserved for the parcel, so at the new radius the resulting velocity of the parcel is:

$$v_p = \frac{\gamma(r)}{r + \delta r}$$

The environment, on the other hand, has:

$$v_{env} = \frac{\gamma(r + \delta r)}{r + \delta r}$$

The inward centripetal acceleration provided by the environmental pressure gradient is $\frac{v_{env}^2}{r+\delta r}$, while the parcel requires $\frac{v_p^2}{r+\delta r}$ to remain at the new radius. So $v_p < v_{env}$ is stable and $v_p > v_{env}$ is unstable. By comparing the velocities above we see this implies that stability requires $\gamma(r) < \gamma(r + dr)$, or:

$$\frac{d\gamma}{dr} > 0$$

Applying to the Rankine vortex, for $r < R$, $\frac{d\gamma}{dr} > 0$ and it is stable. For $r > R$, $\frac{d\gamma}{dr} = 0$, so the system here is neutrally stable. The core of the vortex is stable and the exterior neither amplifies nor damps radial perturbations.

Problem B1: Radiant Bubbles

Hot stars emit vast amounts of ultraviolet photons. These photons ionize the surrounding neutral hydrogen gas, creating a bubble of ionized hydrogen known as a *Strömgren Sphere*.

Consider a star at the center of a uniform cloud of hydrogen with mass density ρ_H. The star emits ionizing photons at a constant rate Q_* (photons/s). Assume the ionized region is in a steady state where the total ionization rate equals the total recombination rate.

Inside the sphere, the recombination rate per unit volume R is:

$$R = n_p n_e \alpha$$

where α is the total volume recombination rate coefficient and n_p and n_e are the respective proton and electron number densities defined as

$$n \equiv \frac{\rho}{m}$$

where m is the particle mass and ρ is the particle density.

(a) Assume the gas inside the sphere is completely ionized and composed of only hydrogen. Derive an expression for the Strömgren radius, r_s, in terms of Q_*, α, the number density of hydrogen n_H, and fundamental constants.

Solution

In steady state, $Q_* = \int R\,dV$. Since the medium is fully ionized, $\rho_e/m_e = \rho_p/m_p = \rho_H/m_H = n_H$ where $m_H \approx m_p$. Then,

$$Q_* = \frac{4\pi}{3} r_s^3 n_H^2 \alpha \implies r_{strom} = \left(\frac{3Q_*}{4\pi\alpha n_H^2}\right)^{1/3}$$

When the star first starts emitting radiation, an ionization front sweeps outward through the surrounding gas as the ionized region expands. In this early stage, assume the gas is initially at rest, the front is a thin discontinuity, and recombinations behind the front can be neglected.

(b) Let s be the speed of the front moving into neutral hydrogen of number density n_H. Let J be the flux of ionizing photons reaching the front (photons per unit area per time).

(i) Derive the expansion speed s in terms of J and n_H.

Solution

Atoms swallowed by front: $N_{at} = n_H(A \cdot s \cdot dt)$. Photons produced to ionize the atoms: $N_{ph} = J \cdot A \cdot dt$. Setting $N_{at} = N_{ph}$ yields $s = J/n_H$.

(ii) Find the radius $r(t)$ of the ionization front as a function of time. Let $r(0) = 0$.

Solution

Substituting J: $s = \frac{Q_*}{4\pi r^2 n_H}$ where $s = \frac{dr}{dt}$. Taking the integral $\int_0^r r^2 dr = \frac{Q_*}{4\pi n_H} t$ and solving for r yields $r(t) = (\frac{3Q_*}{4\pi n_H} t)^{1/3}$.

Inside the sphere, the gas is hot ($T_{in} \approx 10^4$ K), while the neutral gas outside is cold ($T_{out} \approx 100$ K). Initially, the number densities of hydrogen are similar:

$$n_{H,in} \approx n_{H,out}.$$

Assume the gas is ideal, the ionized region remains fully ionized as it expands, and ionization balance is maintained.

(c) (i) Calculate the ratio of the internal pressure P_{in} to external pressure P_{out}.

Solution

$P_{in} = 2n_H k_B T_{in}$ (since ionized gas has two particles per hydrogen atom) and $P_{out} = n_H k_B T_{out}$. Ratio: $\frac{P_{in}}{P_{out}} = \frac{2T_{in}}{T_{out}} = \frac{20000}{100} = 200$.

(ii) At some point the bubble will stop expanding. Find the final radius r in terms of r_s and the temperature ratio (T_{in}/T_{out}).

Solution

At equilibrium, $n_{in} = n_{out}(\frac{T_{out}}{2T_{in}})$. Since $r \propto n^{-2/3}$ from part (a), the final radius $r = r_s(\frac{2T_{in}}{T_{out}})^{2/3}$.

On a larger scale, galaxy clusters are filled with hot gas ($T \approx 10^7$ K) in hydrostatic equilibrium where gravitational forces are balanced by gas pressure:

$$\frac{dP}{dr} = -\rho \frac{d\Phi}{dr},$$

where Φ is the gravitational potential per unit mass. Assume the gas is ideal and approximately isothermal.

(d) (i) If the gravitational potential satisfies

$$\frac{d\Phi}{dr} = \frac{v_c^2}{r}$$

where v_c is constant, show that the gas density follows a power law

$$\rho(r) \propto r^{-\beta}.$$

Find β in terms of v_c, T, and fundamental constants.

Solution

$2k_BT\frac{dn_H}{dr} = -n_Hm_p\frac{v_c^2}{r} \implies \frac{dn_H}{n_H} = -\frac{m_pv_c^2}{2k_BT}\frac{dr}{r}$. Integrating gives $\beta = \frac{m_pv_c^2}{2k_BT}$.

(ii) The gas cools by emitting X-ray photons through free-free emission at a volumetric rate

$$\Lambda \propto n_e^2\sqrt{T}.$$

i. If the cooling time scales as $t_{cool} \propto T^a\rho_e^b$, find a and b.

Solution

The rate of energy loss per time per volume $\Lambda \propto \rho_e\rho_p\sqrt{T} \sim n_e^2\sqrt{T}$. The energy of the medium $E \propto \rho_ek_BT$. Then, $t_{cool} = \frac{E}{\Lambda} \propto \frac{\rho_ek_BT}{\rho_e^2\sqrt{T}} \propto \frac{\sqrt{T}}{\rho_e}$. Thus, $a = 1/2$ and $b = -1$.

ii. Express t_{cool} as a function of r and determine its behavior as as $r \to 0$ and $r \to \infty$.

Solution

Substituting $\rho_e \propto r^{-\beta}$, $t_{cool}(r) \propto \sqrt{T}r^{\beta}$. As $r \to 0$, $t_{cool} \to 0$ (instantaneous cooling at the center of the cluster). As $r \to \infty$, $t_{cool} \to \infty$ (negligible cooling at the outer edges of the cluster).

(iii) From observations of luminous X-rays, we know that the gas at the center of galaxy clusters is hot. Is this expected based on this model? Briefly explain why or why not.

Solution

No, we see that the center of the cluster cools almost instantaneously, however high temperatures are still observed at the center of clusters.

For added context, this means that there has to be an additional mechanism that keeps the center of galaxy clusters hot. This mechanism must be able to produce the equivalent of an entire galaxy's luminosity. Thus, proposed explanations generally require a supermassive black hole which has the gravitational energy budget to produce outflows which can do enough work on the medium to heat it to the observed temperatures. For further reading, see e.g., Fabien (1994) and Hlavacek-Larrondo, Li, and Churazov (2022).

Problem B2: Say Cheese... It's a Conic

You are studying the motion of charged particles constrained to the xy-plane. Particle α, with charge $+1\,\mathrm{C}$, is fixed at the origin.

For motion under an inverse-square central force, trajectories are conic sections with the fixed particle at a focus. In polar coordinates:

- For an attractive interaction (ellipse),

$$r = \frac{r_0}{1 + e\cos\phi}, \qquad e < 1.$$

- For a repulsive interaction (hyperbola),

$$r = \frac{r_0}{e\cos\phi - 1}, \qquad e > 1.$$

You have a camera that takes three snapshots of a moving particle at equal time intervals.

(a) A particle β, with charge $-1\,\mathrm{C}$, moves under the electrostatic force of particle α. In three consecutive snapshots, its positions are

$$(0, -5\,\mathrm{m}), \quad (3\,\mathrm{m}, 0), \quad (0, 5\,\mathrm{m}).$$

Assuming the motion is governed only by the Coulomb interaction with α, determine the maximum distance that particle β reaches from the origin.

Solution

Because the first and third snapshots are at

$$(0, -5\,\mathrm{m}) \qquad \text{and} \qquad (0, 5\,\mathrm{m}),$$

while the second snapshot is at

$$(3\,\mathrm{m}, 0),$$

and the time intervals are equal, the motion is symmetric about the x-axis. Therefore the point $(3, 0)$ must be the point of closest approach, i.e. the *pericenter* of the orbit.

At pericenter,

$$r_{\min} = \frac{r_0}{1+e} = 3.$$

At the two other photographed points, the particle is on the y-axis, so

$$\phi = \pm\frac{\pi}{2}, \qquad r = 5.$$

Since $\cos\big(\pm\pi/2\big) = 0$, we get

$$5 = \frac{r_0}{1 + e \cdot 0} = r_0.$$

So

$$r_0 = 5.$$

Now use the pericenter condition:

$$3 = \frac{5}{1+e} \implies 1 + e = \frac{5}{3} \implies e = \frac{2}{3}.$$

Since $e < 1$, the orbit is an ellipse. The maximum distance occurs at apocenter, where $\phi = \pi$, so

$$r_{\max} = \frac{r_0}{1-e}.$$

Thus

$$r_{\max} = \frac{5}{1 - \frac{2}{3}} = \frac{5}{\frac{1}{3}} = 15\,\text{m}.$$

$$\boxed{15\,\text{m}}$$

(b) Now particle β is replaced by particle γ, which has charge $+2\,\text{C}$ and is free to move in the plane. In three consecutive snapshots, its positions are

$$(3\,\text{m}, -4\,\text{m}), \quad (2\,\text{m}, 0), \quad (3\,\text{m}, 4\,\text{m}).$$

Assuming the motion is governed only by the Coulomb interaction with α, determine the angle θ (measured from the positive x-axis) of the velocity of particle γ at large times.

Solution

Because the first and third snapshots are at

$$(3, -4\,\text{m}) \qquad \text{and} \qquad (3, 4\,\text{m}),$$

while the middle snapshot is at

$$(2, 0),$$

and the time intervals are equal, the motion is symmetric about the x-axis. Thus the point $(2, 0)$ is the point of closest approach.

At closest approach,

$$r_{\min} = \frac{r_0}{e-1} = 2.$$

Hence

$$r_0 = 2(e-1).$$

Now use one of the other photographed points, for example $(3, 4)$. Its distance from the origin is

$$r = \sqrt{3^2 + 4^2} = 5,$$

and

$$\cos\phi = \frac{3}{5}.$$

Therefore

$$5 = \frac{r_0}{e \cdot \frac{3}{5} - 1}.$$

Substitute $r_0 = 2(e-1)$:

$$5 = \frac{2(e-1)}{\frac{3e}{5} - 1}.$$

Multiply through:

$$5\left(\frac{3e}{5} - 1\right) = 2(e-1),$$

$$3e - 5 = 2e - 2,$$

$$e = 3.$$

Then

$$r_0 = 2(e-1) = 4.$$

For a hyperbola, the outgoing asymptote is reached when the denominator goes to zero:

$$e\cos\phi - 1 = 0.$$

Thus

$$\cos\phi = \frac{1}{e} = \frac{1}{3}.$$

So the direction of motion at large times is

$$\theta = \arccos\left(\frac{1}{3}\right).$$

$$\boxed{\theta = \arccos\left(\frac{1}{3}\right) \approx 70.5^\circ.}$$

(c) A family of particles, each identical to particle γ (that is, each has charge $+2\,\mathrm{C}$), approaches from infinity in the xy-plane. All particles have the same speed v_∞ far from the origin and move along lines parallel to the initial asymptotic direction of particle γ from part (2). The particles are injected one at a time, so they do not interact with one another.

For each trajectory, define the *impact parameter* B to be the perpendicular distance between the initial straight-line path of the particle and the origin.

As a particle passes near the scattering center, its direction changes due to Coulomb repulsion. Let α denote the total deflection angle of the trajectory, i.e., the angle between the incoming and outgoing asymptotic directions.

(i) Using the conic form of the trajectory, derive a formula for $B(\alpha)$.

(ii) Using the result of part (2), express $B(\alpha)$ in terms of r_0.

Solution

For a repulsive inverse-square force, the trajectory is a hyperbola

$$r = \frac{r_0}{e\cos\phi - 1}, \qquad e > 1.$$

The asymptotes occur when the denominator vanishes:

$$e\cos\phi - 1 = 0 \implies \cos\phi = \frac{1}{e}.$$

Thus the angle between the asymptotes is

$$\alpha = 2\phi = 2\arccos\left(\frac{1}{e}\right).$$

Hence

$$e = \frac{1}{\cos(\alpha/2)}.$$

Now relate the impact parameter B to the conic parameters. For a hyperbolic trajectory,

$$B = \frac{r_0}{\sqrt{e^2-1}}.$$

Substitute $e = 1/\cos(\alpha/2)$:

$$e^2 - 1 = \frac{1}{\cos^2(\alpha/2)} - 1 = \tan^2(\alpha/2).$$

Thus

$$\sqrt{e^2-1} = \tan(\alpha/2),$$

and therefore

$$B = \frac{r_0}{\tan(\alpha/2)}.$$

$$\boxed{B(\alpha) = r_0\cot\left(\frac{\alpha}{2}\right)}$$

Problem B3: The Trojan Cylinder

In alternating-current systems, the quality factor Q can be defined as

$$Q = 2\pi \times \frac{\text{maximum energy stored}}{\text{energy lost per cycle}}.$$

(a) A circuit consists of an inductor of inductance L and resistance R, carrying a current

$$I(t) = I_0 \cos(\omega t).$$

Using the energy definition of Q, express Q in terms of ω, L, and R.

Solution

The maximum stored energy is

$$U_{\max} = \frac{1}{2} L I_0^2.$$

The average power dissipated is

$$\langle P \rangle = \frac{1}{2} R I_0^2.$$

The energy lost per cycle is

$$E_{\text{loss}} = \langle P \rangle \frac{2\pi}{\omega} = \frac{\pi R I_0^2}{\omega}.$$

Thus,

$$Q = 2\pi \frac{U_{\max}}{E_{\text{loss}}} = \frac{\omega L}{R}.$$

(b) A long solenoid of inductance L is made of superconducting wire and is tightly wound around a thin conducting cylindrical shell of radius a, thickness $t \ll a$, conductivity σ, and length ℓ. Ignore end effects and skin effects.

The current in the solenoid is

$$I(t) = I_0 \cos(\omega t).$$

Using the definition of Q, determine Q in terms of ω, σ, t, a, and L.

Solution

The magnetic field inside the solenoid is

$$B = \mu_0 \frac{N}{\ell} I.$$

The magnetic flux through the cylinder is

$$\Phi = B\pi a^2 = \mu_0 \frac{N}{\ell} \pi a^2 I.$$

Thus, the induced emf around the cylinder is

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\mu_0 \frac{N}{\ell} \pi a^2 \frac{dI}{dt}.$$

Since

$$\frac{dI}{dt} = -\omega I_0 \sin(\omega t),$$

the amplitude of the emf is

$$\mathcal{E}_0 = \mu_0 \frac{N}{\ell} \pi a^2 \omega I_0.$$

The resistance of the cylindrical shell is

$$R_{\text{cyl}} = \frac{2\pi a}{\sigma t \ell}.$$

Thus, the induced current amplitude is

$$I_{\text{ind},0} = \frac{\mathcal{E}_0}{R_{\text{cyl}}} = \frac{\mu_0 \frac{N}{\ell} \pi a^2 \omega I_0}{\frac{2\pi a}{\sigma t \ell}} = \frac{\mu_0 N \sigma t a \omega I_0}{2}.$$

The average power dissipated is

$$\langle P \rangle = \frac{1}{2} R_{\text{cyl}} I_{\text{ind},0}^2.$$

Substituting,

$$\langle P \rangle = \frac{1}{2} \cdot \frac{2\pi a}{\sigma t \ell} \cdot \left(\frac{\mu_0 N \sigma t a \omega I_0}{2} \right)^2.$$

Simplifying,

$$\langle P \rangle = \frac{\pi \mu_0^2 N^2 \sigma t a^3 \omega^2 I_0^2}{4\ell}.$$

The energy lost per cycle is

$$E_{\text{loss}} = \langle P \rangle \frac{2\pi}{\omega} = \frac{\pi^2 \mu_0^2 N^2 \sigma t a^3 \omega I_0^2}{2\ell}.$$

The maximum stored energy is

$$U_{\max} = \frac{1}{2} L I_0^2.$$

Thus,

$$Q = 2\pi \frac{U_{\max}}{E_{\text{loss}}} = 2\pi \frac{\frac{1}{2} L I_0^2}{\frac{\pi^2 \mu_0^2 N^2 \sigma t a^3 \omega I_0^2}{2\ell}} = \frac{2L\ell}{\pi \mu_0^2 N^2 \sigma t a^3 \omega} = \frac{2}{\mu_0 \sigma t a \omega}.$$

(c) Using your result from part (1), determine the effective resistance of the system.

Solution

From part (1),

$$Q = \frac{\omega L}{R_{\text{eff}}},$$

so

$$R_{\text{eff}} = \frac{\omega L}{Q}.$$

Substituting the expression for Q,

$$R_{\text{eff}} = \omega L \cdot \frac{\mu_0 \sigma t a \omega}{2} = \frac{1}{2}\mu_0 \sigma t a L \omega^2.$$

(d) Now suppose the solenoid is connected in series with an external resistor R. Determine the total quality factor of the system in the presence of the conducting cylinder.

Solution

$$Q = \frac{\omega L}{R + R_{\text{eff}}}.$$

(e) Compare the frequency dependence of the quality factor Q for:

- an RL circuit consisting of an ideal inductor L in series with a resistor R,
- the same circuit with the conducting cylinder inserted.

Explain physically why the behavior differs.

Solution

For a standard RL circuit,

$$Q = \frac{\omega L}{R},$$

so Q increases linearly with frequency.

With the conducting cylinder inserted,

$$Q = \frac{\omega L}{R + R_{\text{eff}}}, \qquad R_{\text{eff}} = \frac{1}{2}\mu_0 \sigma t a \, L \, \omega^2.$$

Therefore,

$$Q = \frac{\omega L}{R + \frac{1}{2}\mu_0 \sigma t a \, L \, \omega^2}.$$

Thus:

- at low frequency, $R_{\text{eff}} \ll R$, so
$$Q \approx \frac{\omega L}{R} \propto \omega;$$
- at high frequency, $R_{\text{eff}} \gg R$, so
$$Q \approx \frac{\omega L}{R_{\text{eff}}} \propto \frac{1}{\omega}.$$

So the conducting cylinder causes the quality factor to reach a maximum and then decrease. Physically, this happens because rapidly changing magnetic fields induce strong eddy currents in the cylinder, leading to energy dissipation that grows like ω^2.