AAPT UNITED STATES PHYSICS TEAM
AIP 2026

2026 $F = ma$ Exam

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout, unless otherwise specified.
- You may write in this question booklet and the scratch paper provided by the proctor.
- This test has 25 multiple choice questions. Select the best response to each question, and use a No.2 pencil to completely fill the box corresponding to your choice. If you change an answer, completely erase the previous mark. Only use the boxes numbered 1 through 25 on the answer sheet.
- All questions are equally weighted, but are not necessarily equally difficult.
- You will receive one point for each correct answer, and zero points for each incorrect or blank answer. There is no additional penalty for incorrect answers.
- You may use a hand-held calculator. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any external references, such as books or formula sheets.
- The question booklet, answer sheet and scratch paper will be collected at the end of this exam.
- **To maintain exam security, do not communicate any information about the questions or their solutions until after February 13, 2026.**

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We acknowledge The US Physics Team coaches and other contributors to this year's exams:

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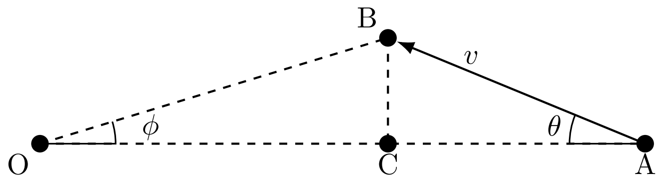
1. In astronomy, some galactic objects appear to sweep across the sky faster than light speed, c. This effect, called superluminal motion, comes purely from geometry and the finite travel time of light, and it has nothing to do with special relativity.

   A jet moves from point A to B at speed $v = \beta c$. The jet emits a pulse of light at A, and a second pulse a time δt later at B. An observer sees these pulses at point O. The angle between the jet and the line of sight is θ. Assume the angle ϕ is small, so the distances from O to points B and C can be treated as equal.



   Find the apparent transverse velocity, v_T, along CB as measured by the observer, in terms of β and θ. Express your answer as $\beta_T \equiv \frac{v_T}{c}$

   A $\beta_T = \frac{\beta \sin\theta}{1-\beta\cos\theta}$ (B) $\beta_T = \beta \sin\theta(1 - \beta\cos\theta)$ (C) $\beta_T = \frac{\beta\sin\theta}{1+\beta\cos\theta}$
   (D) $\beta_T = \frac{\beta\sin\theta}{\sqrt{1-\beta^2}}$ (E) $\beta_T = \beta\tan\theta$

Let $OB = OC = d$. We have:

$$\delta t = t_2 - t_1 \tag{1}$$
$$t_1' = t_1 + \frac{d + v\delta t\cos\theta}{c} \tag{2}$$
$$t_2' = t_2 + \frac{d}{c} \tag{3}$$
$$\delta t' = t_2' - t_1' = \delta t - \frac{v\delta t\cos\theta}{c} \tag{4}$$
$$\implies \beta_T = \frac{v_T}{c} = \frac{1}{c}\frac{v\delta t\sin\theta}{\delta t'} = \frac{\beta\sin\theta}{1-\beta\cos\theta} \tag{5}$$

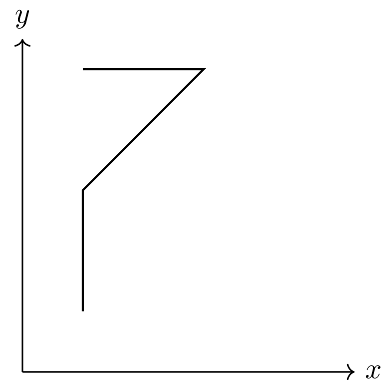
2. A series of dominoes are stood upright in a line. The dominoes have mass density ρ, height h, and spacing d between them. When the first domino is knocked over, a wave of falling dominoes propagates down the line with speed v.

   Now, a second set of dominoes is set up identically, but with all dimensions (h and d) scaled by λ. The mass density is the same. Let v' denote the wave speed of the scaled system. Which of the following best describes how v' depends on λ?

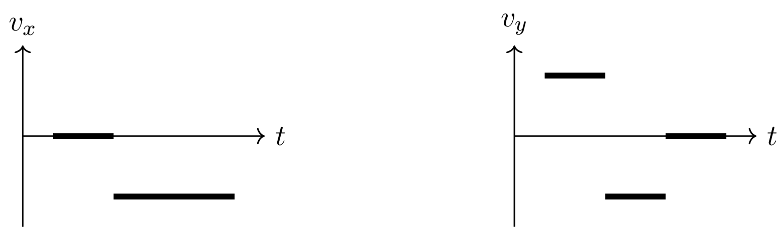
   (A) $v' \sim \frac{1}{\sqrt{\lambda}}$
   (B) $v' \sim \lambda$
   (C) $v' \sim \frac{1}{\lambda}$
   D $v' \sim \sqrt{\lambda}$
   (E) v' is independent of λ

The wave speed should go like the ratio of spacing d to the timescale τ of a domino falling. Clearly τ must go like $\sqrt{\frac{h}{g}}$ by dimensional analysis. So $v \sim d\sqrt{\frac{g}{h}}$. After scaling, the new wave speed must go like: $v_\lambda \sim \lambda d\sqrt{\frac{g}{\lambda h}} \implies v' \sim \lambda^{1/2}$

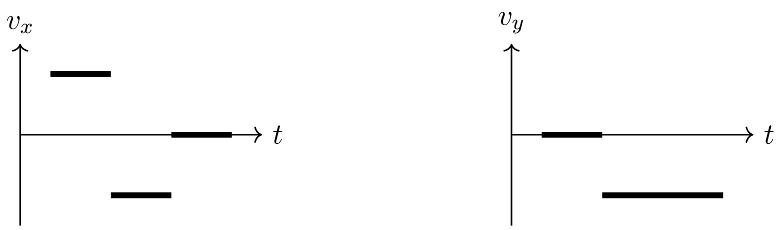
3. A point moves in the xy plane, and its trajectory is shown in the figure below.



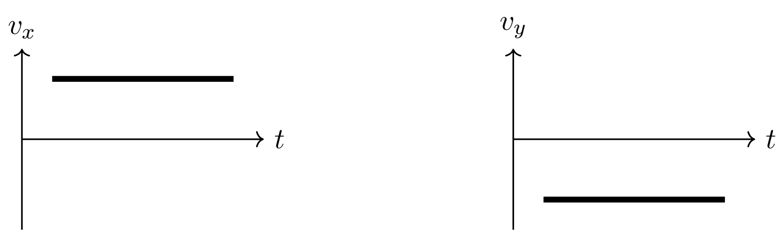
One of the following *pairs* of graphs shows the time dependence of v_x and v_y for this motion. Select the correct pair.



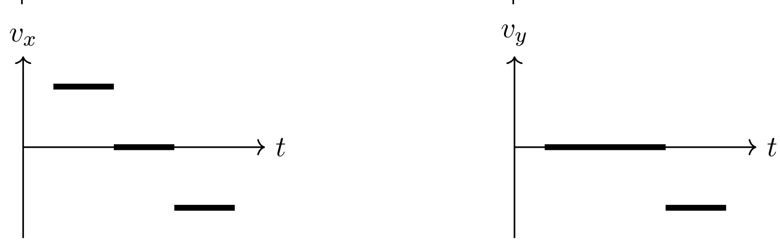
(A)



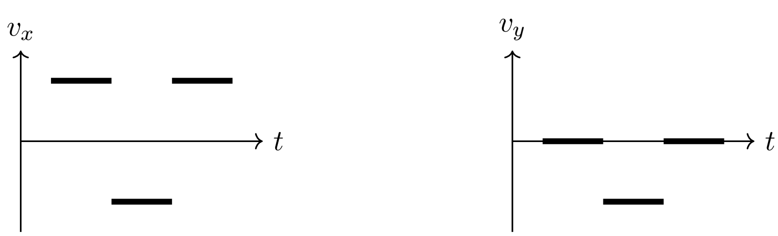
B



(C)



(D)



(E)

If we start from top, then on first segment $v_x > 0$ and $v_y = 0$, next $v_x < 0$ and $v_y < 0$ and on the last segment $v_x = 0$ and $v_y < 0$.

4. Two small balls are launched simultaneously from the same point at some height above horizontal ground. One ball is launched vertically upward with speed 3 m/s, while the other is launched horizontally with speed 4 m/s and lands on the ground at a horizontal distance of 20 m from the launch point. Neglect air resistance. At the moment the second ball lands, how far is it from the first ball?

(A) 25 m (B) 36 m (C) 60 m (D) 80 m (E) 240 m

Their relative velocity magnitude is 5 m/s and it takes 5 seconds to cover 20 m with horizontal component of velocity of 4 m/s

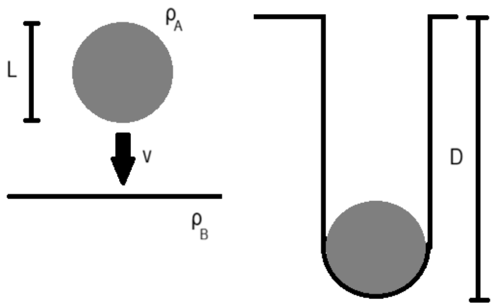
5. A helicopter flies at constant height h along a *circular* trajectory of radius R with constant angular velocity ω, continuously dropping sand. Ignore air resistance. What is the radius of the curve traced by the sand on the ground?

(A) R (B) $R\sqrt{1+h^2}$ (C) $R\sqrt{1+2h^2}$ (D) $R\sqrt{1+\frac{h\omega^2}{g}}$ **E** $R\sqrt{1+\frac{2h\omega^2}{g}}$

Each grain of sand falls for a time $T = \sqrt{2h/g}$ while retaining the helicopter's instantaneous horizontal velocity ωR. During this time it travels a horizontal distance ωRT, perpendicular to the instantaneous radius. The landing point therefore lies at distance

$$R_{\text{sand}} = \sqrt{R^2 + (\omega RT)^2} = R\sqrt{1+\frac{2h\omega^2}{g}}.$$

6. Newton's Impact Depth Approximation provides an estimate on how far a projectile will penetrate a material at high speeds. If a projectile of length L and density ρ_A impacts a wall of density ρ_B, Newton's approximation states that the projectile will penetrate to a depth of $D = L\frac{\rho_A}{\rho_B}$. Assuming this is true for all velocities, how does the average force the wall exerts on the projectile F scale with the initial projectile speed v?



(A) $F \propto v^{1/2}$ (B) $F \propto v$ (C) $F \propto v^{3/2}$ **D** $F \propto v^2$ (E) $F \propto v^{5/2}$

The energy of the projectile scales as v^2, so since the depth of impact is constant. The force the wall exerts on the wall must also scale as v^2

7. The James Webb telescope lies at a point called the Lagrange Point 2 or L2. This is a point where the telescope would remain stationary in a frame rotating with Earth's orbit around the sun, so a telescope placed exactly at L2 would be in an equilibrium position relative to Earth's orbit. What is the stability of this equilibrium point with respect to radial and tangential perturbations?

(A) Stable to both radial and tangential perturbations.

(B) Stable to radial perturbations. Unstable to tangential perturbations.

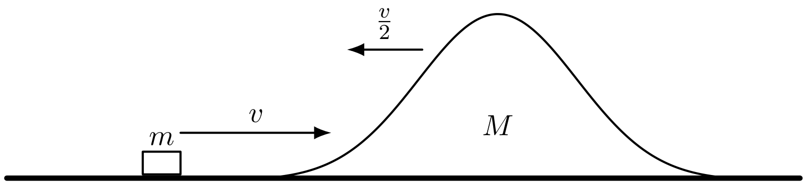
C Stable to tangential perturbations. Unstable to radial perturbations.

(D) Unstable to both radial and tangential perturbations.

(E) None of the listed. L2 is not a real equilibrium point because the centrifugal force is a fictitious force.

Moving an object radially outwards will increase the centrifugal force and lower the gravitational force, so the point is unstable in the radial direction. The tangential force from Earth would increase faster than the tangential force from the centrifugal force, so the object experiences a restoring force tangentially.

8. A puck of mass m moves with speed v toward a heavy bump of mass M, where $M \gg m$. Both the puck and the bump slide without friction. The bump moves toward the puck with speed $\frac{v}{2}$ and provides a smooth transition from the horizontal surface onto the bump. The bump is tall enough that the puck slides back down. Find the maximum height h the puck reaches.



(A) $\frac{v^2}{2g}$ **B** $\frac{9v^2}{8g}$ (C) $\frac{3v^2}{8g}$ (D) $\frac{7v^2}{4g}$ (E) $\frac{3v^2}{4g}$

Writing conservation law in given frame under assumption that bump does not change speed leads to a wrong answer $\frac{3v^2}{8g}$. This assumption works in the bump frame.

9. A vertical cylindrical pipe with a sealed bottom contains no air and is tall enough that its top is open to outer space (no atmosphere). A heavy plate is mounted at the bottom of the cylinder and vibrates vertically. A small ball inside the pipe undergoes elastic collisions with the plate.

The plate vibrates in such a way that, during each collision, it is moving upward with a constant speed equal to $\frac{v}{1000}$, where v is the orbital speed at zero altitude above the Earth. The plate moves downward sufficiently fast that the ball collides with the plate only while the plate is moving upward. The ball is initially dropped from rest from a height equal to the Earth's radius above the plate.

After how many collisions with the plate will the ball escape the Earth's gravitational field?

(A) 207 **B** 208 (C) 414 (D) 415 (E) 1000

At the first impact, the ball's speed equals the escape speed from the Earth. In each subsequent elastic collision with the upward-moving plate, the ball's speed increases by twice the speed of the plate. Therefore, after N collisions the total increase in speed is $2N(v/1000)$.

Requiring the ball's speed to reach the escape speed gives

$$2N\frac{v}{1000} = v(\sqrt{2} - 1),$$

so

$$N = 500(\sqrt{2} - 1) \approx 207.1.$$

After 207 collisions the ball still returns, but after the 208th collision it escapes the Earth's gravitational field. Hence, the required number of collisions is $\boxed{208}$.

10. Consider the following three solid objects, each of mass M and uniform density:
(i) a half-ball of radius a;
(ii) a cylinder of radius a and height a;
(iii) a cylinder of radius a and height $2a$.

For each object, consider the gravitational acceleration at the center of its flat circular face. Let g_{ball}, g_a, and g_{2a} denote these accelerations for the half-ball, the cylinder of height a, and the cylinder of height $2a$, respectively. Which of the following is correct?

(A) $g_a > g_{\text{ball}} > g_{2a}$ (B) $g_{\text{ball}} > g_{2a} > g_a$ **C** $g_{\text{ball}} > g_a > g_{2a}$
(D) $g_{2a} > g_{\text{ball}} > g_a$ (E) $g_{2a} > g_a > g_{\text{ball}}$

In all three cases, symmetry ensures that the gravitational acceleration at the specified point is perpendicular to the flat circular face.

Starting from the half-ball, imagine continuously deforming it into a cylinder of radius a and height a while keeping the total mass fixed. During this deformation, mass elements move farther from the observation point and spread farther from the symmetry axis, reducing the perpendicular component of the gravitational field. Thus,

$$g_{\text{ball}} > g_a.$$

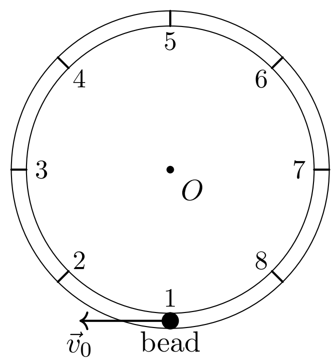
Next, deform the cylinder of height a into a cylinder of the same radius but height $2a$. This moves mass farther away along the symmetry axis and further reduces the gravitational acceleration at the center of the base:

$$g_a > g_{2a}.$$

Therefore,

$$g_{\text{ball}} > g_a > g_{2a}.$$

11. A circular track of mass m lies flat horizontally on a frictionless table and is free to move on it. A bead also of mass m can slide without friction on the track. The track is initially at rest, and it is labeled every 45° with tick marks. The bead is placed on tick 1 and is given an initial velocity $\vec{v}_0$ tangential to the track as shown below.



What is the speed of the track's center O when the bead reaches tick 3?

(A) v_0 (B) $\frac{v_0}{2}$ (C) $\frac{\sqrt{3}v_0}{2}$ **D** $\frac{v_0}{\sqrt{2}}$ (E) $\frac{v_0}{2\sqrt{2}}$

Both momentum and energy are conserved in the system. Since the track is frictionless no torque is applied at any point. Take the track's velocity as $\vec{u}$ and the bead's as $\vec{v}$. We have four unknown vector components, and we can use the conserved quantities and the track constraint to solve:

$$p_x = \text{const.} \implies v_0 = v_x + u_x$$
$$p_y = 0 \implies u_y = -v_y$$
$$E = \text{const.} \implies u_x^2 + u_y^2 + v_x^2 + v_y^2 = v_0^2$$
$$\text{constraint:} \implies v_x = u_x$$

Solving for $u = \sqrt{u_x^2 + u_y^2}$ gives $u = \frac{v_0}{\sqrt{2}}$.

12. A wake surfer of total mass M (surfer plus board) is being towed at a constant horizontal velocity v across a flat lake. The wakeboard has a specific geometry such that, at this speed, it is partially submerged and provides a static buoyant force F_B (where $F_B < Mg$).

To support the remainder of the weight, the board moves with an angle of attack θ relative to the horizontal water surface. Assume that the water exerts a reaction force strictly normal to the bottom surface of the board. Which of the following expressions represents the horizontal tension T in the tow rope required to maintain this constant velocity?

(A) $T = Mg\tan\theta$
(B) $T = (Mg - F_B)\tan\theta$
(C) $T = (Mg - F_B)\sin\theta$
(D) $T = Mg\sin\theta$
(E) $T = F_B\cos\theta$

The surfer is moving at a constant velocity, meaning the net force is zero in both the vertical and horizontal directions.

Lets first consider the vertical direction. The weight of the surfer and board is supported by two upward forces: the static buoyant force (F_B) and the vertical component of the hydrodynamic reaction force from the water (R_y), that is:

$$\Sigma F_y = 0 \implies F_B + R_y = Mg$$

Therefore the vertical lift required from the hydro-planing action:

$$R_y = Mg - F_B$$

As stated, the water exerts a force strictly normal to the bottom surface of the board. Since the board is tilted at an angle of attack θ relative to the horizontal, the normal force vector $\vec{R}$ is tilted backward by the angle θ relative to the vertical axis.

Using vector decomposition, the horizontal component (R_x, (drag)) and the vertical component (R_y, (lift)) are related by the tangent of the angle:

$$\tan\theta = \frac{R_x}{R_y}$$

Through simple trigonometry we get:

$$R_x = R_y\tan\theta$$

Now, the surfer is being towed at a constant speed, so the forward tension (T) in the rope must balance the drag component (R_x).

$$T = R_x$$

Substituting the expression for the drag we get:

$$T = R_y\tan\theta$$

Now, substitute the expression for the lift ($R_y = Mg - F_B$) above we get:

$$T = (Mg - F_B)\tan\theta$$

13. A block of mass m slides along a surface with a coefficient of kinetic friction μ_k. The block is confined to move between two walls separated by a distance L. Attached to each wall is an ideal, perfectly elastic spring with a very large force constant k.

The block is launched from the midpoint between the walls with an initial speed v_0 directed toward one of the springs. It bounces back and forth between the springs, losing speed only due to friction with the surface.

Assume the time spent in contact with the springs is negligible compared to the travel time between them. Which of the following expressions correctly gives the total time T required for the block to come to a permanent rest?

(A) $T = \frac{v_0}{2\mu_k g L}$

(B) $T = \frac{v_0}{\mu_k g}\left(1 - e^{-\frac{\mu_k g}{v_0}}\right)$

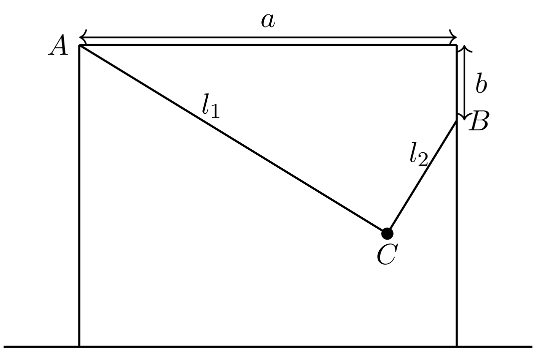
C $T = \frac{v_0}{\mu_k g}$

(D) $T = \frac{v_o}{\mu_k g L}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$

(E) $T = \sqrt{\frac{2L}{\mu_k g}}$

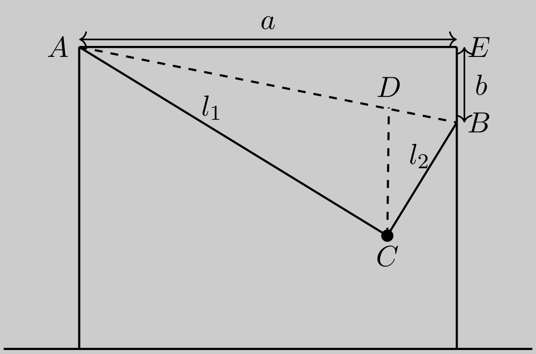
Considering that the force of friction always opposes the direction of travel, and taking into account that the block is bouncing back and forth in a perfectly elastic collision (with the ”bounce” time being negligible), we can recognize that the mass bouncing back and forth is equivalent to just a mass traveling along a flat surface with some coefficient of kinetic friction mu_k. Therefore, the answer is just c

14. In the following swing, all lines are rigid rods and their lengths satisfy the relation $l_1^2 + l_2^2 = a^2 + b^2$. What is the period of small oscillation of the mass at point C?



(A) $2\pi\sqrt{\frac{l_1^2+l_2^2}{ga}}$ **B** $2\pi\sqrt{\frac{l_1 l_2}{ga}}$ (C) $2\pi\sqrt{\frac{l_1 l_2}{g\sqrt{a^2+b^2}}}$ (D) $2\pi\sqrt{\frac{a l_1 l_2}{g(l_1^2+l_2^2)}}$ (E) $2\pi\sqrt{\frac{b l_1 l_2}{g(l_1^2+l_2^2)}}$

The effective point of suspension can be chosen to be any point on the line segment connecting points A and B, and the effective gravity will be actual gravity projected along the effective pendulum rod. In particular, if we choose the point D just above C, the effective gravity is equal to the actual gravity.



To find the length from C to the line AB, we use the law of sines: $\sin(\angle CAD)/|CD| = \sin(\angle ADC)/l_1$. Now, notice that $\sin(\angle ADC) = \sin(\angle ABE) = a/\sqrt{a^2+b^2}$, and $\sin(\angle CAD) = \sin(\angle CAB) = l_2/\sqrt{a^2+b^2}$ (since ABC is a right triangle). Simplifying, $|CD| = l_1 l_2/a$.

15. A ball launcher fires balls along the floor at some initial speed, applying no rotation to them. The balls initially slip along the floor, then start rolling without slipping. Ignore the potential deformation of the ball and flooring during this process, as well as air resistance. How does the final speed of the rolling ball depend on the coefficient of friction μ between the ball and the floor?

(A) The final speed is larger when μ is large

B The final speed is the same regardless of μ

(C) The final speed is larger when μ is small

(D) The final speed is larger when μ is small for high launch speeds, and when μ is large for low launch speeds

(E) The final speed is larger when μ is large for high launch speeds, and when μ is small for low launch speeds

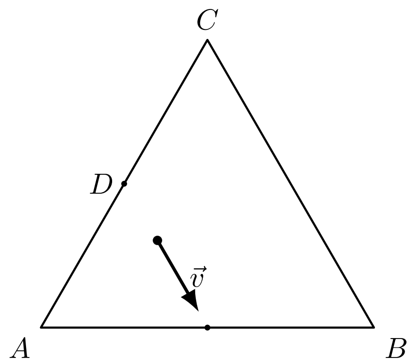
While the ball is slipping, friction is acting on it: the frictional force will be μmg. This generates a linear acceleration $\dot{v} = -\mu g$, and an angular acceleration of $\dot{\omega} = \mu mgr/I$ (where r is the radius and I is the angular moment of inertia of the ball). The ball will stop slipping when $v = \omega r$; the value of v when that occurs depends only on properties of the ball, not on μ.

16. A small puck of mass m slides without friction inside a rigid, uniform *triangle rack* of mass $5m$, whose inner boundary is an equilateral triangle ABC. The rack is initially at rest and can move freely on a frictionless horizontal table. All collisions between the puck and the rack are perfectly elastic.

At some instant, the puck strikes the midpoint of side AB from inside the rack. Just before this collision, the puck's velocity is parallel to side BC.

After the collision at the midpoint of AB, the puck collides with the rack again, and then collides once more. At which part of the triangle does this second collision after the initial one occur?

(Some answer choices refer to the midpoint D of side AC.)



(A) a point in the interior of segment AB

(B) a point in the interior of segment BC

(C) a point in the interior of segment AD

(D) a point in the interior of segment DC

E the point D

During each impact, the contact force produces zero net torque on the rack about its center of mass, so the triangle does not rotate. For an elastic collision between frictionless bodies, the tangential component of the relative velocity at the point of contact is unchanged, while the normal component reverses sign *without changing its magnitude*, independent of the masses of the colliding objects. Thus, in the rack's instantaneous rest frame, the collision is equivalent to specular reflection from a frictionless wall. After the first collision at the midpoint of AB, the puck next collides with the midpoint of BC, and the subsequent collision asked for in the problem occurs at point D.

17. A ladder is leaning against a vertical wall. The ladder is a uniform rod of mass M and length L, and both the wall and the ground are frictionless. The ladder is released from rest from an almost-vertical position and begins to slide. What is the speed of the point of the ladder that is in contact with the floor when it is a horizontal distance $\frac{\sqrt{3}L}{2}$ away from the wall?

(A) $0.46\sqrt{gL}$ (B) $0.51\sqrt{gL}$ (C) $0.56\sqrt{gL}$ **D** $0.61\sqrt{gL}$ **E** $0.66\sqrt{gL}$

Key physical subtlety: the ladder *loses contact with the wall* before the base reaches $x = \frac{\sqrt{3}}{2}L$. Thus there are two distinct phases: (i) motion while touching both wall and floor, and (ii) motion after detachment. The multiple-choice options $0.61\sqrt{gL}$ and $0.66\sqrt{gL}$ correspond to the two most natural "no-calculus" estimates at the transition.

Phase I (still touching wall): detachment speed. Let θ be the angle the ladder makes with the horizontal. While both contacts are maintained, the constraints give

$$x_B = L\cos\theta, \qquad y_{\text{CM}} = \frac{L}{2}\sin\theta.$$

Starting from rest near vertical, the drop in the center of mass from $\theta = 90^\circ$ to angle θ is

$$\Delta U = Mg\left(\frac{L}{2} - \frac{L}{2}\sin\theta\right) = \frac{1}{2}MgL(1-\sin\theta).$$

Using the instantaneous center (intersection of normals at the contacts), the kinetic energy can be written

$$T = \frac{1}{2}I_P\Omega^2, \qquad I_P = I_{\text{CM}} + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2,$$

and the base speed is $v_B = \Omega(L\sin\theta)$. Energy conservation gives

$$\frac{1}{2}\left(\frac{1}{3}ML^2\right)\Omega^2 = \frac{1}{2}MgL(1-\sin\theta) \quad\Longrightarrow\quad \Omega^2 = \frac{3g}{L}(1-\sin\theta),$$

so

$$v_B^2 = \Omega^2L^2\sin^2\theta = 3gL(1-\sin\theta)\sin^2\theta.$$

The ladder detaches when the normal force at the wall drops to zero; the standard result for this setup is

$$\sin\theta_{\text{det}} = \frac{2}{3}.$$

Substituting into the expression above gives

$$v_{B,\text{det}}^2 = 3gL\left(1-\frac{2}{3}\right)\left(\frac{2}{3}\right)^2 = \frac{4}{9}gL \quad\Rightarrow\quad v_{B,\text{det}} = \frac{2}{3}\sqrt{gL} \approx 0.67\sqrt{gL}.$$

Thus the choice $0.66\sqrt{gL}$ matches the *exact* detachment speed.

What about the asked position $x = \frac{\sqrt{3}}{2}L$? If one (incorrectly) assumes the ladder *remains in contact with the wall* all the way to $x = \frac{\sqrt{3}}{2}L$, then $\cos\theta = \frac{\sqrt{3}}{2}$, i.e. $\theta = 30^\circ$, and the same formula yields

$$v_B^2 = 3gL(1-\tfrac{1}{2})(\tfrac{1}{2})^2 = \frac{3}{8}gL \quad\Rightarrow\quad v_B = \sqrt{\frac{3gL}{8}} \approx 0.61\sqrt{gL},$$

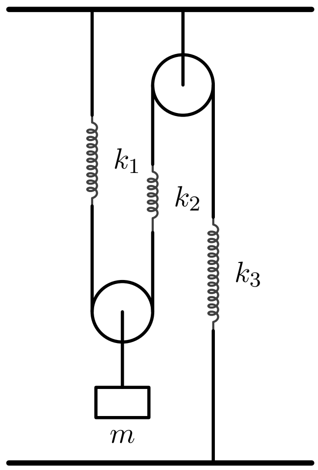
which is exactly the $0.61\sqrt{gL}$ choice.

Why we accept both 0.61 and 0.66. The correct motion *does* detach earlier, and determining the speed *at* $x = \frac{\sqrt{3}}{2}L$ requires analyzing the post-detachment evolution (translation + rotation) over time. That level of modeling is beyond what can be resolved reliably by a short algebraic argument in the F=ma format. However:

- $0.66\sqrt{gL}$ is an *exact* milestone speed (at detachment),
- $0.61\sqrt{gL}$ is the natural value obtained under the common "stays in contact" simplification,
- distinguishing which is closer to the true value at $x = \frac{\sqrt{3}}{2}L$ requires post-detachment dynamics not intended for this exam.

Therefore, we award full credit for either choice $0.61\sqrt{gL}$ or $0.66\sqrt{gL}$.

18. Three springs and a block of mass m are connected as shown in the figure below. The spring constants are $k_1 = k$, $k_2 = k/5$, and $k_3 = k/3$. All pulleys and ropes are massless, and there is no friction anywhere in the system. Find the ratio of the period of oscillations of this system, T, to the period T_0 of a simple mass–spring system with mass m and spring constant k. The value of $\frac{T}{T_0}$ is:



The value of $\frac{T}{T_0}$ is:

(A) $\frac{2}{3}$ **B** $\frac{3}{2}$ (C) $\frac{\sqrt{2}}{\sqrt{3}}$ (D) $\frac{\sqrt{7}}{6}$ (E) $\frac{\sqrt{6}}{7}$

The displacement of mass m is y. It is related to extension of springs. Assume that x_1 is expansion of spring with the spring constant k_1, x_2 correspond to spring with k_2 and x_3 to the spring with k_3. These extensions and displacement are related by $y = \frac{x_1+x_2+x_3}{2}$. Also $k_1x_1 = k_2x_2 = k_3x_3$. Equation of motion now looks like

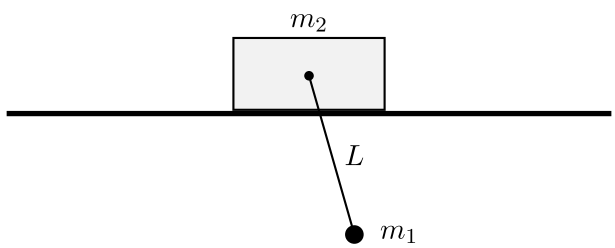
$$ma_{1x}\frac{1 + \frac{k_1}{k_2} + \frac{k_1}{k_3}}{2} = -2k_1x_1 + mg.$$

From this equation, after substitution all k_i, we get the angular frequency

$$\omega^2 = \frac{4}{9}\frac{k}{m}$$

and $T/T_0 = \frac{3}{2}$

19. A point mass m_1 is attached to a box of mass m_2 by a massless, inextensible rope of fixed length L, attached to the *center* of the box. The box can slide without friction on a horizontal surface. The mass m_1 swings freely under gravity, and *all motion occurs in the plane of the picture.* Neglect air resistance. Find the angular frequency ω of small oscillations about the stable equilibrium.



In case you need the small-angle approximation, you may use

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + \cdots, \qquad \sin\theta = \theta - \frac{1}{6}\theta^3 + \cdots.$$

(A) $\omega = \sqrt{\frac{m_1+m_2}{m_2}\frac{g}{L}}$ (B) $\omega = \sqrt{\frac{m_1+m_2}{m_1}\frac{g}{L}}$ (C) $\omega = \sqrt{\frac{m_1}{m_1+m_2}\frac{g}{L}}$

(D) $\omega = \sqrt{\frac{m_2}{m_1+m_2}\frac{g}{L}}$ (E) $\omega = \sqrt{\frac{g}{L}}$

For small oscillations, it is convenient to work in an *inertial* frame in which the *total horizontal momentum of the system is zero.* Since there are no external horizontal forces, the frame is inertial:

$$0 = m_2 v + m_1\big(v + L\Omega\cos\theta\big) \approx m_2 v + m_1(v + L\Omega),$$

where v is the horizontal velocity of the box and $\Omega = \dot{\theta}$. Thus,

$$v = -\frac{m_1}{m_1+m_2}L\Omega.$$

The kinetic energy (to quadratic order) is

$$T \approx \frac{1}{2}m_2 v^2 + \frac{1}{2}m_1(v+L\Omega)^2 = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}L^2\Omega^2 \equiv \frac{1}{2}A\Omega^2.$$

The potential energy of the pendulum mass is

$$U = -m_1 g L\cos\theta \approx \text{const} + \frac{1}{2}m_1 g L\,\theta^2 \equiv \frac{1}{2}B\theta^2,$$

where the constant term does not affect the dynamics.

Thus the energy has the form

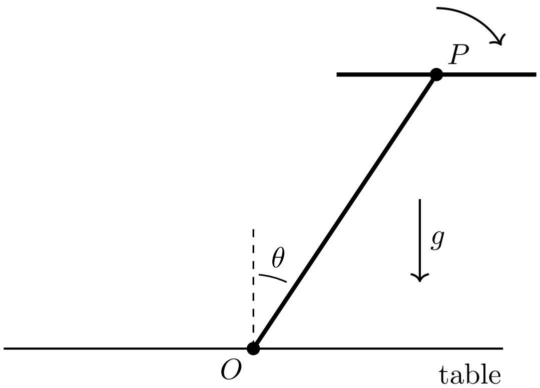
$$E = \frac{1}{2}A\Omega^2 + \frac{1}{2}B\theta^2,$$

which is equivalent to a simple harmonic oscillator. Therefore,

$$\omega = \sqrt{\frac{B}{A}} = \sqrt{\frac{m_1+m_2}{m_2}\frac{g}{L}}.$$

20. Two uniform rigid rods lie in a vertical plane. The lower rod rests on a rough horizontal table at a single point O and makes angle $\theta \neq 0$ with the vertical. The rod can pivot about point O without slipping.

The second rod is rigidly attached at its center of mass to the top of the lower rod at point P. A motor causes the second rod to rotate in the plane about P. The motor is rigidly attached to the lower rod and does not exert forces or torques on the table.



Which of the following types of motion of the second rod could keep the angle θ fixed in time?

(A) The second rod rotates with constant angular velocity

(B) The second rod rotates with angular velocity that varies periodically in time

(C) The system cannot remain at constant θ because only internal torques are produced by the motor

(D) The second rod rotates with angular velocity that increases linearly in time

(E) θ changes because the moment of inertia of the second rod about point O varies during the motion

To keep the angle fixed, the gravitational torque about O must be balanced by an opposite dL_O/dt produced by the motor, which occurs via clockwise angular acceleration of the second rod. Since the second rod is connected at center of mass, the gravitational torque is independent of the second rod's orientation, so dL_O/dt should be constant, thus we require constant angular acceleration (D). Constant angular velocity would produce zero dL_O/dt. The varying moment of inertia of the second rod about O is not relevant.

21. A student stands on a large horizontal merry-go-round ($R = 2.0\,\text{m}$) at an initial radius of $r_0 = 1.0\,\text{m}$. Both rotate at constant angular speed $\omega = 1.2$ rad/s. The student wants to get off without walking and performs a sequence of identical vertical jumps of height $h = 0.31\,\text{m}$. Ignore air resistance.

What is the minimum number of jumps needed for the student to land off the platform? You may assume that the merry-go-round is much more massive than the student and that friction instantly brings the student back into co-rotation with the platform.

(A) 3

(B) 4

(C) 5

(D) 6

(E) It is impossible to move outward by purely vertical jumps

First determine the time of flight for one jump. From vertical kinematics, a jump that reaches a maximum height h has a total time in the air

$$\tau = 2\sqrt{\frac{2h}{g}}.$$

Now analyze the horizontal motion in the inertial frame. At the instant of takeoff from radius r, the student has horizontal tangential speed

$$v = \omega r,$$

because they were co-rotating with the platform. While airborne, there is no horizontal force, so this tangential speed remains constant.

Choose coordinates so that takeoff occurs from

$$(x, y) = (r, 0)$$

with velocity along the positive y-direction. After a time τ,

$$x = r, \qquad y = v\tau = \omega r\tau.$$

The distance from the center when the student lands is therefore

$$r' = \sqrt{r^2 + (\omega r\tau)^2} = r\sqrt{1 + \omega^2\tau^2}.$$

When the student lands, friction brings them back into co-rotation with the platform at this new radius before the next jump. Thus, each jump multiplies the radius by a factor

$$F = \sqrt{1 + \omega^2\tau^2}.$$

After N jumps, the radius is approximately

$$r_N \approx r_0 F^N.$$

To reach the edge, require

$$r_N \geq 2.0 \text{ m}.$$

This gives

$$F^N \geq 2 \quad \Rightarrow \quad N > \frac{\ln 2}{\ln F} \approx 4.5.$$

The smallest integer satisfying this condition is

$$N = 5.$$

22. A tricycle of mass $m = 100\,\text{kg}$ is traveling north on a horizontal surface. The geometry of the tricycle is defined as follows:

- Wheelbase (distance from front axle to rear axle): $L = 1.0\,\text{m}$
- Rear track width (distance between rear wheels): $d = 1.0\,\text{m}$
- Center of mass (CM) location: on the longitudinal symmetry axis, a distance $b = 0.4\,\text{m}$ forward of the rear axle
- Radius of gyration about the CM: $k = 0.5\,\text{m}$

(The radius of gyration k is defined by $I_{\text{CM}} = mk^2$, where I_{CM} is the moment of inertia of the tricycle about a vertical axis through its center of mass.)

Seeing a patch of ice on their right side ahead, the rider panics and slams on the brakes. The front wheel and the left rear wheel slide on dry pavement with a coefficient of kinetic friction $\mu_k = 0.5$, while the right rear wheel slides on frictionless ice ($\mu = 0$).

This asymmetry in friction forces produces a net torque about the center of mass, causing the tricycle to begin rotating. Calculate the initial angular acceleration α of the tricycle.

A 3.0 rad/s^2, Counter-Clockwise
(B) 3.0 rad/s^2, Clockwise
(C) 3.3 rad/s^2, Counter-Clockwise
(D) 6.0 rad/s^2, Counter-Clockwise
(E) 3.3 rad/s^2 Clockwise

Consider the forces acting on the three wheels while breaking. The two wheels in contact with the rough surface generates a kinetic friction force opposing the motion (pointing South). Using a symmetry argument we can reduce this down to just the left wheel, which is displaced a lateral distance $d/2$ to the left of the central axis.

Now we calculate the torque generated by the left rear wheel about the CM. The position vector $\vec{r}$ points to the left (West), and the force vector $\vec{F}$ points backward (South). Using the cross product we gett:

$$\vec{\tau}_{left} = (r\hat{i}) \times (-F\hat{j})$$

The force pulls backward on the left side of the vehicle. This creates a twisting moment that rotates the vehicle counter-clockwise (Left).

$$\tau_{net} = F_{fric} \cdot \frac{d}{2} \quad \text{(Counter-Clockwise)}$$

Since the torque is non-zero and counter-clockwise, the tricycle acquires an initial angular acceleration α in the counter-clockwise direction.

$$\alpha \propto \tau_{net} > 0 \implies \text{CCW Rotation}$$

To find the exact value, we first calculate the moment of inertia (I) using the radius of gyration k:

$$I = mk^2 = 100(0.5)^2 = 25\ \text{kg} \cdot \text{m}^2$$

Next, we determine the normal forces on the wheels to find the friction magnitude. Summing moments about the rear axle allows us to find the normal force on the front wheel (N_f):

$$\Sigma\tau_{rear} = 0 \implies N_f L = mgb$$

Plugging in our values above therefore gives us:

$$N_f(1.0) = (100)(10)(0.4) \implies N_f = 400 \text{ N}$$

The remaining weight is supported by the two rear wheels. Since the CM is on the symmetry axis, this load is split equally:

$$N_{rear\_total} = mg - N_f = 1000 - 400 = 600 \text{ N}$$

$$N_{left} = \frac{600}{2} = 300 \text{ N}$$

Now we calculate the kinetic friction force on the left rear wheel:

$$F_{friction} = \mu_k N_{left} = 0.5(300) = 150 \text{ N}$$

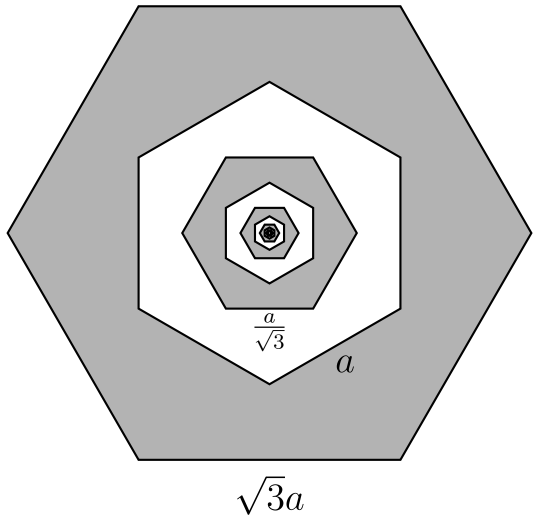
As we stated previously, the net torque is generated solely by the left rear wheel acting at a lever arm of $d/2$:

$$\tau_{net} = F_{friction} \times \frac{d}{2} = 150 \times 0.5 = 75 \text{ N} \cdot \text{m}$$

Finally, from Newton's second law for rotation we get:

$$75 = 25\alpha \implies \alpha = 3.0 \text{ rad/s}^2$$

23. Consider the infinitely repeating hexagonal sequence where the largest hexagon has side length $\sqrt{3}a$ and is filled with mass. Inside it, a smaller hexagon of side length a is rotated 30° clockwise and cut out. Then, an even smaller hexagon of side length $a/\sqrt{3}$ is rotated again by 30° clockwise and filled. This pattern continues infinitely. Let I_1 be the moment of inertia of this 2D figure about an axis through its center of mass, perpendicular to its plane and I_2 be the moment of inertia of a regular hexagon with side length $\sqrt{3}a$. What is I_1/I_2 equal to? ($\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$)



(A) $\frac{1}{4}$ (B) $\frac{2\sqrt{3}+3}{9\sqrt{3}+9}$ **C** $\frac{9}{10}$ (D) $\frac{3\sqrt{3}+6}{9\sqrt{3}+6}$ (E) none of the listed

Assuming moment of inertia for hexagon for side length a is $I = kma^2 \propto k\rho a^4$ since mass $m \propto \rho a^2$ (uniform mass yay). Then, we see

$$I_2 = k\rho(\sqrt{3}a)^4 = 9k\rho a^4$$

For I_1, it is just an infinite sum

$$I_1 = k\rho(\sqrt{3}a)^4 - k\rho(a^4) + k\rho(a/\sqrt{3})^4 + k\rho(a/\sqrt{3}^2)^4 + \cdots = 9k\rho a^4 - k\rho a^4 \sum_{n=0}^{\infty} \left(-\left(\frac{1}{\sqrt{3}}\right)^4 \right)^n$$

$$= 9k\rho a^4 - k\rho a^4 \sum_{n=0}^{\infty} \left(-\frac{1}{9}\right)^n = 9k\rho a^4 - k\rho a^4 \frac{1}{1+\frac{1}{9}} = 9k\rho a^4 - \frac{9}{10}k\rho a^4 = \frac{81}{10}k\rho a^4$$

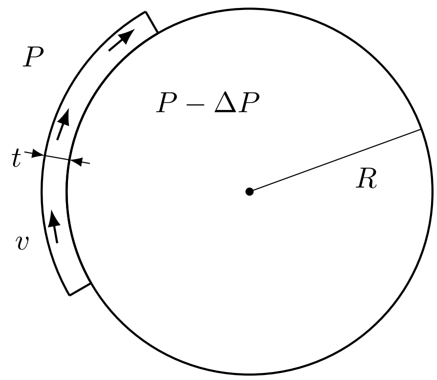
Thus

$$\frac{I_1}{I_2} = \frac{\frac{81}{10}k\rho a^4}{9k\rho a^4} = \frac{9}{10}$$

24. A thin sheet of air with density ρ is blown at a uniform speed v toward a smooth, convex cylindrical surface of radius R. The sheet has uniform thickness $t \ll R$ and remains everywhere tangent to the cylinder while it stays attached. Neglect gravity and viscosity.

The pressure on the outer side of the air sheet is constant and equal to P. The pressure on the side adjacent to the cylinder is lower by an amount ΔP. The air sheet follows the curvature of the cylinder until it can no longer remain attached.

Given the parameters below, what is the *maximum integer speed* $v_{\max}$ for which the sheet can remain attached to the surface?



$$\rho = 1.2\ \text{kg/m}^3, \qquad t = 5.0\ \text{mm}, \qquad R = 5.0\ \text{cm}, \qquad \Delta P_{\max} = 120\ \text{Pa}.$$

(A) 8 m/s (B) 22 m/s **(C)** 31 m/s (D) 44 m/s (E) 60 m/s

Consider a jet element that subtends an angle $\delta\theta$ on the cylinder and ignore any motion not occurring in the plane of the page. The element's arc length per unit width is

$$\delta s = R\,\delta\theta.$$

The mass of this slice is

$$\delta m = \rho t\,\delta s = \rho t\,(R\,\delta\theta).$$

To follow the curved surface, the jet must undergo centripetal acceleration v^2/R. Thus the required inward force per unit width is

$$\delta F_{\text{req}} = \delta m\,\frac{v^2}{R} = \rho t\,(R\,\delta\theta)\,\frac{v^2}{R} = \rho t\,v^2\,\delta\theta.$$

The pressure difference ΔP acts over the area

$$A = R\,\delta\theta,$$

giving inward force

$$dF_p = \Delta P\,(R\,\delta\theta).$$

Set $\delta F_p = \delta F_{\text{req}}$:

$$\Delta P\,(R\,\delta\theta) = \rho t\,v^2\,\delta\theta.$$

Cancel $\delta\theta$ and solve:

$$\boxed{\Delta P = \frac{\rho\, t\, v^2}{R}}.$$

Attachment requires $\Delta P \le \Delta P_{\text{max}}$, so

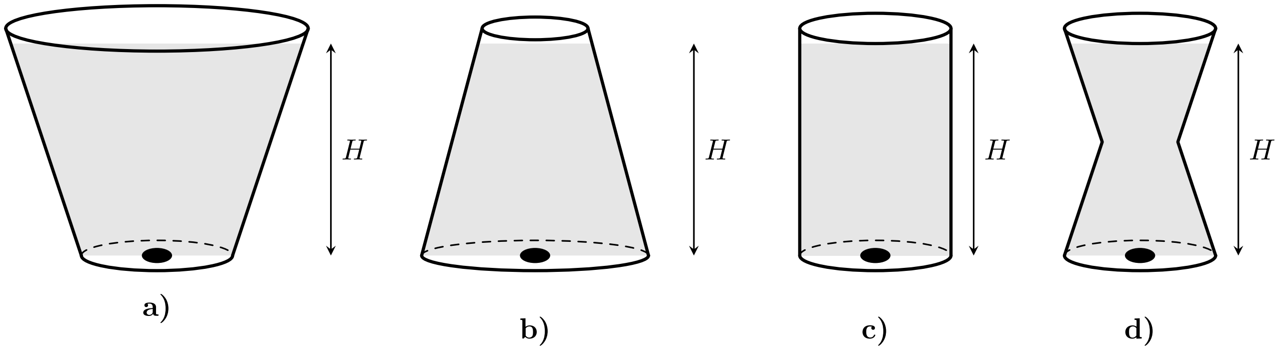
$$\Delta P_{\text{max}} = \frac{\rho\, t\, v^2}{R} \quad \Rightarrow \quad v^2 = \frac{\Delta P_{\text{max}} R}{\rho t}.$$

Plugging in the numbers:

$$v \approx 31.6 \text{ m/s}.$$

$$\boxed{v_{\text{max}} \approx 31 \text{ m/s}}$$

25. The five vases below are filled with water to the same initial volume and initial height H. Each vase has an identical drainage hole at the bottom.



Which vase will drain completely in the shortest amount of time?

A Vase a

(B) Vase b

(C) Vase c

(D) Vase d

(E) All vases drain in the same amount of time.

The rate at which water exits the container is given by Torricelli's Law:

$$v(z) = \sqrt{2gz}$$

The rate at which the volume of water V decreases is the product of the flux and the hole area a:

$$\frac{dV}{dt} = -av(z) = -a\sqrt{2gz}$$

Implicitly we can recognize that there are only one dynamic variable on each side of the equation, namely V and z and effectively reduce this to:

$$\frac{dV}{dt} \propto \frac{dz}{dt}$$

To get an exact solution we would need to integrate, but this is not necessarily required and we can instead use our intuition to solve it. That is, we can divide up each volume into separate slices and recognize that total time to drain the vase is the sum of the time required to drain each individual packet:

$$T_{total} = \sum_{i=1}^{N} \Delta t_i$$

The time Δt required to drain a specific packet is inversely proportional to the flow rate at that moment:

$$\Delta t \approx \frac{\Delta V}{FlowRate} \propto \frac{1}{\sqrt{z}}$$

From this we can recognize that a slice of water drains faster if it is positioned at a high height z than if it is positioned near the bottom. To minimize the total time, we must arrange the geometry of the vase such that the majority of the water volume is located where the drainage is fastest (i.e. at high z). Taking this into account we see that the answer must be a.